On the Study of Forward Kolmogorov System and the Corresponding Problems for Inhomogeneous Continuous-Time Markov Chains

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Abstract An inhomogeneous continuous-time Markov chain $X(t)$ with finite or countable state space under some natural additional assumptions is considered. As a consequence, we study a number of problems for the corresponding forward Kolmogorov system, which is the linear system of differential equations with special structure of the matrix $A(t)$. In the countable situation we have an equation in the space of sequences l_1. The important properties of $X(t)$ (such as weak and strong ergodicity, perturbation bounds, truncation bounds) are closely connected with behaviour of the solutions of the forward Kolmogorov system as $t \to \infty$. The main problems and some approaches for their solution are discussed in the paper.

Keywords Forward Kolmogorov system · Markov chains

1 Introduction

Continuous-time Markov chains are widely used for the study of stochastic models in the natural and technical sciences, such as queuing theory, biology, chemistry, etc.

Let $\{X(t),\ t \geq 0\}$ be a continuous-time Markov chain with state space $\mathscr{X} = \{0, 1, 2 \dots\}$. Denote by $p_{ij}(s,t) = P\{X(t) = j\,|X(s) = i\}$, $i, j \geq 0$, $0 \leq s \leq t$ the transition probabilities of $X(t)$ and by $p_i(t) = P\{X(t) = i\}$ – the probability that the Markov chain $X(t)$ is in state i at time t. Let $\mathbf{p}(t) = (p_0(t), p_1(t), \dots)^T$ be the probability distribution vector at instant t. Throughout the paper we assume that in an element of time h the possible transitions and their associated probabilities are

$$p_{ij}(t, t+h) = \begin{cases} q_{ij}(t)h + \alpha_{ij}(t,h), & \text{if } j \neq i \\ 1 + q_{ii}(t)h + \alpha_i(t,h), & \text{if } j = i, \end{cases} \quad i, j \in \mathscr{X}, \tag{1}$$

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where all the $\alpha_i(t,h)$ are $o(h)$ uniformly in i, i.e. $\sup_i |\alpha_i(t,h)| = o(h)$ and

$$q_{ii}(t) = -\sum_{k\in\mathscr{X}, k\neq i} q_{ik}(t).$$

The matrix $Q(t) = (q_{ij}(t))_{i,j=0}^{\infty}$ is called the intensity (or infinitesimal) matrix of the chain $\{X(t),\ t \ge 0\}$.

The Markov chain $X(t)$ is called homogeneous if Q is a constant matrix, and it is called inhomogeneous in the opposite case.

As a rule, in the inhomogeneous case we will assume that the intensity functions $q_{ij}(t)$ are locally integrable on the interval $[0, \infty)$.

Henceforth it is assumed that the $Q(t)$ is essentially bounded, i.e.

$$\sup_i |q_{ii}(t)| = L(t) \le L < \infty, \tag{2}$$

for almost all $t \ge 0$.

In many problems, condition (2) can be weakened and replaced by $L(t) < \infty$, for almost all $t \ge 0$.

Then the probabilistic dynamics of the process $\{X(t),\ t \ge 0\}$ is given by the forward Kolmogorov system

$$\frac{d}{dt}\mathbf{p}(t) = A(t)\mathbf{p}(t), \tag{3}$$

where $A(t) = Q^T(t)$ is the transposed intensity matrix. All column sums of this matrix are zeros for any $t \ge 0$, and $A(t)$ is essentially nonnegative (i.e. all its off-diagonal elements are nonnegative for any $t \ge 0$).

Throughout the paper by $\|\cdot\|$ we denote the l_1-norm, i. e. $\|\mathbf{p}(t)\| = \sum_{k\in\mathscr{X}} |p_k(t)|$, and $\|Q(t)\| = \sup_{j\in\mathscr{X}} \sum_{i\in\mathscr{X}} |q_{ij}|$. Let Σ be a set all stochastic vectors, i. e. l_1 vectors with non-negative coordinates and unit norm. Hence we have $\|A(t)\| = 2\sup_{k\in\mathscr{X}} |q_{kk}(t)| \le 2L$ for almost all $t \ge 0$. Hence the operator function $A(t)$ from l_1 into itself is bounded for almost all $t \ge 0$ and locally integrable on $[0;\infty)$. Therefore we can consider (3) as a differential equation in the space l_1 with bounded operator.

It is well known (see [2]) that the Cauchy problem for differential Eq. (3) has a unique solutions for an arbitrary initial condition, and $\mathbf{p}(s) \in \Sigma$ implies $\mathbf{p}(t) \in \Sigma$ for $t \ge s \ge 0$.

Denote by $E(t,k) = E(X(t)|X(0) = k)$ the conditional expected number of 'particles' in the system at instant t, provided that initially (at instant $t = 0$) k 'particles were present in the system.

In order to obtain perturbation bounds we consider a class of perturbed Markov chains $\{\bar{X}(t), t \ge 0\}$ defined on the same state space $\mathscr{X}$ as the original Markov chain $\{X(t), t \ge 0\}$, with the intensity matrix $\bar{A}(t)$ and the same restrictions as imposed on $A(t)$. It is assumed that $\|\hat{A}(t)\| = \|A(t) - \bar{A}(t)\| \le \varepsilon$, for almost all $t \ge 0$, which means the perturbations are considered to be small.

Before proceeding to the derivation of the main results of the paper, we recall two definitions. Recall that a Markov chain $\{X(t),\ t \geq 0\}$ is called *weakly ergodic*, if $\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \to 0$ as $t \to \infty$ for any initial conditions $\mathbf{p}^*(0)$ and $\mathbf{p}^{**}(0)$, where $\mathbf{p}^*(t)$ and $\mathbf{p}^{**}(t)$ are the corresponding solutions of (3). A Markov chain $\{X(t),\ t \geq 0\}$ has the limiting mean $\varphi(t)$, if $\lim_{t\to\infty} (\varphi(t) - E(t,k)) = 0$ for any k.

It is clear, therefore, that the study of the qualitative properties and the derivation of estimates for Markov chains with continuous time is reduced to the study of the corresponding properties of solutions of the forward Kolmogorov (3) system on Σ.

A general approach to obtaining sharp bounds on the rate of convergence via the notion of the logarithmic norm of an operator function was recently discussed in detail in our papers [34–36]. The first studies in this direction were published since 1980-s for birth-death models, see [25, 26]. In [34, 35] we have highlighted four fairly broad classes of finite and countable Markov chains, for which the forward Kolmogorov system can be transformed into a system with an essentially nonnegative matrix. Moreover, it turns out that similar results can be obtained for some other models, see, for example [37]. Computation of the limiting characteristics for such chains using bounds on the rate of convergence and truncations technique introduced in [30, 33].

The approach is based on studying the norm of the Cauchy operator of the reduced forward Kolmogorov system by estimation of the so-called logarithmic norm of an operator function. The method of the complete study of the process $X(t)$ that describes the number of claims in the system assumes the construction of a) upper bounds for the rate of convergence of the limit mode, providing that, beginning from a certain time, say, t^*, the probability characteristics of the process $X(t)$ do not depend on the initial conditions (up to a given discrepancy); b) analogous lower bounds which are also very important and provide that the “independence” of the initial conditions cannot appear before a certain time, say, t_*; c) stability bounds providing that if the structure of the matrix of intensities of the process is taken into account in an appropriate way, and the errors in intensities are small, then the basic characteristics of the process are calculated in an adequate way; d) approximations to the process by means of truncation by similar processes with a lesser number of states and construction of the corresponding estimates for the error. Finally, applying the results of a), c), d) to the system with 1-time-periodic intensities and solving the forward Kolmogorov system with the simplest initial condition $X(0) = 0$ for the truncated process on the interval $[t^*, t^* + 1]$, as a result we obtain all basic probability characteristics of both the process $X(t)$, and close “perturbed” processes. Note that the item a) is most important, because after the corresponding bounds are obtained, the solutions of other problems can be constructed automatically on the base of the results of [27–34].

Generally speaking, instead of obtaining the solution to the Cauchy problem on a short time interval by some methods that are approximate anyway, which does not provide actual information of the real basic properties of the system, we determine the time interval, on which the Cauchy problem for the forward Kolmogorov system must really be solved and find this solution.

It is worth noting that exact estimates of the rate of convergence yield exact estimates of stability (perturbation bounds), see [8, 11, 14, 15, 17, 23, 32] and references therein. Moreover, such connections and their significance were highlighted in the recent communication by Mitrophanov, see

http://alexmitr.com/talk_DDE2018_Mitrophanov_FIN_post_sm.pdf.

The approach is based on the special properties of linear systems of differential equations with essentially nonnegative matrices. Specifically, if the column-wise sums of the elements of this matrix are identical and equal to, say, $-\alpha^*(t)$, then the exact upper bound of order $\exp\left\{-\int_0^t \alpha^*(u)\,du\right\}$ can be obtained for the rate of convergence of the solutions of the system in the corresponding metric. Moreover, if the column-wise sums of the absolute values of the elements of this matrix are identical and equal to, say, $\chi^*(t)$, then the exact lower bound of order $\exp\left\{-\int_0^t \chi^*(u)\,du\right\}$ can be obtained for the convergence rate as well. The bounds are obtained in three steps. At first step one excludes the (0) state from the forward Kolmogorov system of differential equations and thus obtains the new system with the new intensity matrix which is, in general, not non-diagonally non-negative. The second step is to transform the new intensity matrix in such a way that non-diagonally elements are non-negative and which leads to (loosely speaking) least distance between specifically defined upper and lower bounds. At third step one uses the logarithmic norm for the estimation of the convergence rate.

Here the key step is the second one. The transformation is made using a sequence of positive numbers $\{d_i, i \geq 1\}$, which does not have any probabilistic meaning and can be considered as an analogue of Lyapunov functions.

The advantages of this three-step approach is that it allows one to deal with time-homogeneous and time-inhomogeneous processes and it leads to exact both upper and lower bounds for the convergence rate. In time-homogeneous case the approach allows one to obtain the corresponding bounds for the decay parameter and gives an explicit bounds in total variation norm.

2 General Transformations

Recall that one has introduced $A(t)$ as the transposed intensity matrix $Q(t)$. Thus it has the form

$$A(t) = \begin{pmatrix} a_{00}(t) & a_{01}(t) & \cdots & a_{0r}(t) & \cdots \\ a_{10}(t) & a_{11}(t) & \cdots & a_{1r}(t) & \cdots \\ a_{20}(t) & a_{21}(t) & \cdots & a_{2r}(t) & \cdots \\ \cdots \\ a_{r0}(t) & a_{r1}(t) & \cdots & a_{rr}(t) & \cdots \\ \cdots \end{pmatrix}, \tag{4}$$

where $a_{ii}(t) = -\sum_{k\in\mathcal{X},k\neq i} a_{ki}(t)$. Since $p_0(t) = 1 - \sum_{i=1}^{\infty} p_i(t)$ due to normalization condition, one can rewrite the system (3) as follows:

$$\frac{d}{dt}\mathbf{z}(t) = B(t)\mathbf{z}(t) + \mathbf{f}(t), \tag{5}$$

where

$$\mathbf{f}(t) = (a_{10}(t), a_{20}(t), \dots)^T, \ \mathbf{z}(t) = (p_1(t), p_2(t), \dots)^T,$$

$$B(t)=\begin{pmatrix} a_{11}-a_{10} & a_{12}-a_{10} & \cdots & a_{1r}-a_{10} & \cdots \\ a_{21}-a_{20} & a_{22}-a_{20} & \cdots & a_{2r}-a_{20} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{r1}-a_{r0} & a_{r2}-a_{r0} & \cdots & a_{rr}-a_{r0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{6}$$

Each entry of B depends on t. See detailed discussion of this transformation in [7, 27].

There is the following simple relationship between pairs, $\mathbf{z}^{(i)} = \mathbf{z}^{(i)}(t),\ t \ge 0,\ i = 1, 2$, of solutions of (5) and pairs of solutions of (3), $\mathbf{p}^{(i)} = \mathbf{p}^{(i)}(t),\ t \ge 0,\ i = 1, 2$:

$$\begin{aligned} \left\|\mathbf{p}^{(1)} - \mathbf{p}^{(2)}\right\|_1 &= \left|p_0^{(1)} - p_0^{(2)}\right| + \sum_{i\ge1}\left|p_i^{(1)} - p_i^{(2)}\right| \\ &= \left|1 - \sum_{i\ge1} p_i^{(1)} - \left(1 - \sum_{i\ge1} p_i^{(2)}\right)\right| + \left\|\mathbf{z}^{(1)} - \mathbf{z}^{(2)}\right\|_1 \\ &= \left|\sum_{i\ge1}\left(p_i^{(2)} - p_i^{(1)}\right)\right| + \left\|\mathbf{z}^{(1)} - \mathbf{z}^{(2)}\right\|_1 \le \sum_{i\ge1}\left|p_i^{(2)} - p_i^{(1)}\right| + \left\|\mathbf{z}^{(1)} - \mathbf{z}^{(2)}\right\|_1 \\ &= 2\left\|\mathbf{z}^{(1)} - \mathbf{z}^{(2)}\right\|_1, \quad t \ge 0. \end{aligned}$$

Consequently,

$$\left\|\mathbf{z}^{(1)} - \mathbf{z}^{(2)}\right\|_1 \le \left\|\mathbf{p}^{(1)} - \mathbf{p}^{(2)}\right\|_1 \le 2\left\|\mathbf{z}^{(1)} - \mathbf{z}^{(2)}\right\|_1, \quad t \ge 0, \tag{7}$$

which will be used in the study of stability and ergodicity.

Let $\{d_i,\ i \ge 1\}$ with $d_1 = 1$ be an increasing sequence of positive numbers. Put

$$W = \inf_{i\ge1} \frac{d_i}{i}. \tag{8}$$

and denote by D the upper triangular matrix of the following form:

$$D = \begin{pmatrix} d_1 & d_1 & d_1 & \cdots \\ 0 & d_2 & d_2 & \cdots \\ 0 & 0 & d_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{9}$$

Let l_{1D} be the corresponding space of sequences

$$l_{1D} = \left\{\mathbf{z}(t) = (p_1(t), p_2(t), \cdots)^T \mid \|\mathbf{z}(t)\|_{1D} \equiv \|D\mathbf{z}(t)\|_1 < \infty\right\}$$

and introduce also the auxiliary norm $\|\cdot\|_{1E}$ defined as $\|\mathbf{z}(t)\|_{1E} = \sum_{k=1}^{\infty} k|p_k(t)|$. Then in $\|\cdot\|_{1D}$ norm the following two inequalities hold:

$$\begin{aligned}
\|\mathbf{z}(t)\|_{1D} &= d_1 \left|\sum_{i=1}^{\infty} p_i(t)\right| + d_2 \left|\sum_{i=2}^{\infty} p_i(t)\right| \\
&\quad + d_3 \left|\sum_{i=3}^{\infty} p_i(t)\right| + \dots \\
&\geq \left(\left|\sum_{i=1}^{\infty} p_i(t)\right| + \left|\sum_{i=2}^{\infty} p_i(t)\right| + \left|\sum_{i=3}^{\infty} p_i(t)\right| + \dots\right) \\
&\geq \frac{1}{2}\left(\left(\left|\sum_{i=1}^{\infty} p_i(t)\right| + \left|\sum_{i=2}^{\infty} p_i(t)\right|\right)\right. \\
&\quad + \left.\left(\left|\sum_{i=2}^{\infty} p_i(t)\right| + \left|\sum_{i=3}^{\infty} p_i(t)\right|\right) + \dots\right. \\
&\geq \frac{1}{2}\sum_{i=1}^{\infty} |p_i(t)| = \frac{1}{2}\|\mathbf{z}(t)\|_1,
\end{aligned} \tag{10}$$

$$\begin{aligned}
\|\mathbf{z}(t)\|_{1E} &= \sum_{k=1}^{\infty} k|p_k(t)| \\
&= \sum_{k=1}^{\infty} \frac{k}{d_k} d_k |p_k(t)| \leq W^{-1} \sum_{k=1}^{\infty} d_k |p_k(t)| \\
&= W^{-1} \sum_{k=1}^{\infty} d_k \left|\sum_{i=k}^{\infty} p_i(t) - \sum_{i=k-1}^{\infty} p_i(t)\right| \\
&\leq W^{-1} \sum_{k=1}^{\infty} d_k \left(\left|\sum_{i=k}^{\infty} p_i(t)\right| + \left|\sum_{i=k-1}^{\infty} p_i(t)\right|\right) \\
&\leq \frac{2}{W} \sum_{k=1}^{\infty} d_k \left|\sum_{i=k}^{\infty} p_i(t)\right| \leq \frac{2}{W}\|\mathbf{z}(t)\|_{1D}.
\end{aligned} \tag{11}$$

3 Logarithmic Norm and Related Bounds

Recall here the definition of logarithmic norm.

The concept of *logarithmic norm* of a square matrix was developed independently by Dahlquist [1] and Lozinskiĭ [12] as a tool to derive error bounds in the numerical integration of initial-value problems for a system of ordinary differential equations (see also the survey papers [21] and [20]). For the linear differential equation in a Banach space with locally integrable operator function this notion was discussed in [2].

Let $B(t), \ t \geq 0$ be a one-parameter family of bounded linear operators on a Banach space $\mathscr{B}$ and let I denote the identity operator.
For each $t \geq 0$, the number

$$\gamma\left(B(t)\right) = \lim_{h \to +0} \frac{\|I + hB(t)\| - 1}{h} \tag{12}$$

is called the logarithmic norm of the operator $B(t)$.
The logarithmic norm of the matrix $B(t) = \left\{b_{ij}(t)\right\}, \quad t \geq 0$ corresponding to a linear operator on the vector space $\mathscr{B}$ equipped with ℓ_1- norm, is

$$\gamma\left(B(t)\right) = \sup_j \left(b_{jj}(t) + \sum_{i \neq j} \left|b_{ij}(t)\right| \right), \quad t \geq 0. \tag{13}$$

Associate now the family of operators $B(t), \quad t \geq 0$ with the system of differential equations

$$\frac{d\mathbf{x}}{dt} = B(t)\,\mathbf{x}, \quad t \geq 0, \tag{14}$$

where the functions $b_{ij}(t), \ 0 \leq i, j < \infty$ are assumed to be locally integrable on $[0, \infty)$, and denote by $V(t, s), \ 0 \leq s \leq t$ the corresponding Cauchy operator (hence $\mathbf{x}(t) = V(t, s)\mathbf{x}(s)$ for any $0 \leq s \leq t$). Then the logarithmic norm of the operator $B(t)$ is related to $V(t, s), \ 0 \leq s \leq t$ by

$$\gamma\left(B(t)\right) = \lim_{h \to +0} \frac{\|V(t+h, t)\| - 1}{h}, \quad t \geq 0. \tag{15}$$

From the latter one can deduce the following bounds on the $\mathscr{B}$-norm of the Cauchy operator $V(t, s), \quad 0 \leq s \leq t$:

$$e^{-\int_s^t \gamma(-B(\tau))\, d\tau} \leq \|V(t, s)\| \leq e^{\int_s^t \gamma(B(\tau))\, d\tau}, \quad 0 \leq s \leq t. \tag{16}$$

Moreover, for any solution $\mathbf{x}(t) \in \mathscr{B}, \quad t \geq 0$ of (14) we have

$$\|\mathbf{x}(t)\| \geq e^{-\int_s^t \gamma(-B(\tau))\, d\tau} \|\mathbf{x}(s)\|. \tag{17}$$

We will also make use of the fact that if $\mathscr{B}$ is a vector space with norm ℓ_1 and all diagonal elements of B are non-negative then, by (13)

$$\gamma\left(B\left(t\right)\right)=\sup_{j}\sum_{i}b_{ij}\left(t\right),\quad t\geq 0,$$

and, *a fortiori*, for any solution $\mathbf{x}\left(t\right),\ t\geq 0$ of (14), s.t. $\mathbf{x}\left(s\right)\geq\mathbf{0}$, we have

$$\left\|\mathbf{x}\left(t\right)\right\|\geq e^{\int_{s}^{t}\inf_{j}\sum_{i}b_{ij}(\tau)\,d\tau}\left\|\mathbf{x}\left(s\right)\right\|,\quad 0\leq s\leq t. \tag{18}$$

Consider the Eq. (5) in the space l_{1D}, where $B(t)$ and $\mathbf{f}(t)$ are locally integrable on $[0,+\infty)$. Let one compute the logarithmic norm of operator function $B(t)$.

Then for the logarithmic norm of the operator function $B(t)$ in $\|\cdot\|_{1D}$ norm the following equality holds:

$$\gamma(B(t))_{1D}=\gamma(DB(t)D^{-1})_{1}.$$

Denote by $B^{*}(t)=DB(t)D^{-1}$, and the elements of $B^{*}(t)$ by $b^{*}_{ij}(t)$ i.e. $B^{*}(t)=\left(b^{*}_{ij}(t)\right)_{i,j=1}^{\infty}$. Assume that

$$b^{*}_{ij}(t)\geq 0,\ i\neq j,\ t\geq 0. \tag{19}$$

Remark 1. Note that assumption (19) of essential nonnegativity of the reduced matrix $B^{*}(t)$ is key to the possibility of effective use of the method of the logarithmic norm. In particular, this assumption is fulfilled for four important classes of Markov chains, which we consider in the next section.

Put

$$\alpha_{i}\left(t\right)=-\sum_{j=0}^{\infty}b^{*}_{ji}(t),\quad \chi_{i}\left(t\right)=-\sum_{j=0}^{\infty}|b^{*}_{ji}(t)|,\ i\geq 1, \tag{20}$$

and let $\alpha(t)$ and $\beta(t)$ denote the least lower and the least upper bound of the sequence of functions $\{\alpha_{i}(t),\ i\geq 1\}$ and $\chi(t)$ denote the least upper bound of $\{\chi_{i}(t),\ i\geq 1\}$ i.e.

$$\alpha\left(t\right)=\inf_{i\geq 1}\alpha_{i}\left(t\right),\ \beta\left(t\right)=\sup_{i\geq 1}\alpha_{i}\left(t\right), \tag{21}$$

$$\chi\left(t\right)=\sup_{i\geq 1}\chi_{i}\left(t\right). \tag{22}$$

Then the logarithmic norms of $B(t)$ and $(-B(t))$ are equal to

$$\gamma\left(B\left(t\right)\right)_{1D}=\sup_{i}\alpha_{i}(t)=-\alpha\left(t\right),$$
$$\gamma\left(-B\left(t\right)\right)_{1D}=\sup\chi_{i}\left(t\right)=\chi\left(t\right).$$

If now one defines $\mathbf{v}(t) = D(\mathbf{p}^*(t) - \mathbf{p}^{**}(t))$, then the following equation holds

$$\frac{d}{dt}\mathbf{v}(t) = DB(t)\,D^{-1}\mathbf{v}(t), \tag{23}$$

Notice that due to (19), the inequality $\mathbf{v}(s) \geq \mathbf{0}$ implies that $\mathbf{v}(t) \geq \mathbf{0}$ for any $t \geq s$. Hence

$$\frac{d}{dt}\sum_{i=1}^{\infty} v_i(t) \geq -\beta(t)\sum_{i=1}^{\infty} v_i(t), \tag{24}$$

and one can obtain establish the corresponding bounds on the rate of convergence, perturbation bounds, and estimates on the error of truncations.

4 Four Classes of Markov Chains

These classes were previously studied in [34, 35]. We use here the terminology from Markov chain theory and queueing in parallel depending on context.

Class (I). Inhomogeneous birth-death processes (BDP), where all $a_{ij}(t) = 0$ for any $t \geq 0$ if $|i-j| > 1$, and $a_{i,i+1}(t) = \mu_{i+1}(t),\ \ a_{i+1,i}(t) = \lambda_i(t)$ - birth and death rates respectively. This process, in particular, is a standard model as queue-length process for a general Markovian queue $M_n(t)/M_n(t)/1$.

In this situation we obtain

$$B^*(t) = \begin{pmatrix} -(\lambda_0(t)+\mu_1(t)) & \mu_1(t) & 0 & \cdots & 0 & \cdots & \cdots \\ \lambda_1(t) & -(\lambda_1(t)+\mu_2(t)) & \mu_2(t) & \cdots & 0 & \cdots & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \\ 0 & \cdots & \cdots & \lambda_{r-1}(t) & -\left(\lambda_{r-1}(t)+\mu_r(t)\right) & \mu_r(t) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \tag{25}$$

if $S = \infty$, and

$$B^*(t) = \begin{pmatrix} -(\lambda_0(t)+\mu_1(t)) & \mu_1(t) & 0 & \cdots & 0 \\ \lambda_1(t) & -(\lambda_1(t)+\mu_2(t)) & \mu_2(t) & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & \lambda_{S-1}(t) & -\left(\lambda_{S-1}(t)+\mu_S(t)\right) \end{pmatrix}, \tag{26}$$

if $S < \infty$.

One can see that the transformed matrix $B^*(t)$ is essentially nonnegative for any t, that is all off-diagonal elements of this matrix are nonnegative for any t.

Remark 2. This class is the most studied. It includes, in particular, models of systems of the theory of queues $M_t/M_t/N$, and $M_t/M_t/N/N$, see for instance [3–5, 7, 13, 22, 25–27, 30] and references therein. For the first one, we get the matrix (25) with

$\lambda_k(t) = \lambda(t)$ and $\mu_k(t) = \min(k, N) \cdot \mu(t)$, and for the second one we get (26) with $\lambda_k(t) = \lambda(t)$ and $\mu_k(t) = k\mu(t)$.

Another approach to the study of close models with discrete time was considered in [9].

Class (II). Inhomogeneous queue-length process for a queue with batch arrivals and single services, where $a_{ij}(t) = 0$ for any $t \geq 0$ if $i < j - 1$, all arrival rates do not depend on the size of a queue, where $a_{i+k,i}(t) = a_k(t)$ for $k \geq 1$ - the rate of arrival of a group of k customers, $a_{i,i+1}(t) = \mu_{i+1}(t)$ - the service rate. Such models in simplest situations were firstly considered in [16].

In this situation we have

$$B^*(t) = \begin{pmatrix} a_{11}(t) & \mu_1(t) & 0 & \cdots & 0 \\ a_1(t) & a_{22}(t) & \mu_2(t) & \cdots & 0 \\ a_2(t) & a_1(t) & a_{33}(t) & \mu_3(t) & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{27}$$

if $S = \infty$, and

$$B^*(t) = \begin{pmatrix} a_{11}(t) - a_S(t) & \mu_1(t) & 0 & \cdots & 0 \\ a_1(t) - a_S(t) & a_{22}(t) - a_{S-1}(t) & \mu_2(t) & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ a_{S-1}(t) - a_S(t) & \cdots & \cdots & a_1(t) - a_2(t) & a_{SS}(t) - a_1(t) \end{pmatrix}, \tag{28}$$

if $S < \infty$.

One can see that the transformed matrix $B^*(t)$ is certainly essentially nonnegative for any t if arrival rates $a_k(t)$ are decrease in k.

Class (III). Inhomogeneous queue-length process for the queueing model with batch services and single arrivals, where all $a_{ij}(t) = 0$ for any $t \geq 0$ if $i > j + 1$, and all service rates do not depend on the size of a queue, where $a_{i,i+k}(t) = b_k(t),\ k \geq 1$ is the rate of service of a group of k customers, and $a_{i+1,i}(t) = \lambda_i(t)$ is the arrival rate, see also [16]. One can find more modern studies of these models in [10].

Here we obtain

$$B^*(t) = \begin{pmatrix} -(\lambda_0(t) + b_1(t)) & b_1(t) - b_2(t) & b_2(t) - b_3(t) & \cdots & \cdots & \\ \lambda_1(t) & -\big(\lambda_1(t) + \sum\limits_{i \leq 2} b_i(t)\big) & b_1(t) - b_3(t) & \cdots & \cdots & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \cdots & \cdots & \lambda_{r-1}(t) & -\big(\lambda_{r-1}(t) + \sum\limits_{i \leq r} b_i(t)\big) & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{29}$$

if $S = \infty$, and

$$B^*(t) = \begin{pmatrix} -(\lambda_0(t) + b_1(t)) & b_1(t) - b_2(t) & b_2(t) - b_3(t) & \cdots & b_{S-1}(t) - b_S(t) \\ \lambda_1(t) & -\big(\lambda_1(t) + \sum\limits_{i \le 2} b_i(t)\big) & b_1(t) - b_3(t) & \cdots & b_{S-2}(t) - b_S(t) \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & \lambda_{S-1}(t) & -\big(\lambda_{S-1}(t) + \sum\limits_{i \le S} b_i(t)\big) \end{pmatrix}, \quad (30)$$

if $S < \infty$.

One can see that the transformed matrix $B^*(t)$ is certainly essentially nonnegative for any t if service rates $b_k(t)$ are decrease in k.

Class (IY). Queue-length process for a non-stationary queueing model with batch arrivals and group services, where all rates do not depend on the size of a queue, here $a_{i+k,i}(t) = a_k(t)$, and $a_{i,i+k}(t) = b_k(t)$ for $k \ge 1$ are the rates of arrival and service of a group of k customers respectively. Such process were studied in [18, 19, 31].

$$B^* = \begin{pmatrix} a_{11}(t) & b_1(t) - b_2(t) & b_2(t) - b_3(t) & \cdots & \cdots & \\ a_1(t) & a_{22}(t) & b_1(t) - b_3(t) & \cdots & \cdots & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \\ a_{r-1}(t) & \cdots & \cdots & a_1(t) & a_{rr}(t) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \quad (31)$$

if $S = \infty$, and

$$B^*(t) = \begin{pmatrix} a_{11}(t) - a_S(t) & b_1(t) - b_2(t) & b_2(t) - b_3(t) & \cdots & b_{S-1}(t) - b_S(t) \\ a_1(t) - a_S(t) & a_{22}(t) - a_{S-1}(t) & b_1(t) - b_3(t) & \cdots & b_{S-2}(t) - b_S(t) \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ a_{S-1}(t) - a_S(t) & \cdots & \cdots & a_1(t) - a_2(t) & a_{SS}(t) - a_1(t) \end{pmatrix}, \quad (32)$$

if $S < \infty$.

In this case the transformed matrix $B^*(t)$ is surely essentially nonnegative for any t if all arrival and service rates $a_k(t)$ and $b_k(t)$ are decreasing on k.

5 General Bounds for Continuous-Time Markov Chains

Rate of Convergence

Theorem 1. *Let there exist an increasing sequence $\{d_j,\ j \ge 1\}$ of positive numbers with $d_1 = 1$, such that (19) holds, and $\alpha(t)$ defined by (21) satisfies*

$$\int_0^{\infty} \alpha(t)\, dt = +\infty. \tag{33}$$

Then the Markov chain $\{X(t),\ t \ge 0\}$ *is weakly ergodic and the following bounds hold:*

$$\begin{aligned} & e^{-\int_s^t \chi(u)du} \|\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\|_{1D} \\ & \quad \le \|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1D} \\ & \le e^{-\int_s^t \alpha(u)du} \|\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\|_{1D}, \end{aligned} \tag{34}$$

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \le 4e^{-\int_s^t \alpha(u)du} \|\mathbf{z}^*(s) - \mathbf{z}^{**}(s)\|_{1D}, \tag{35}$$

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1E} \le \frac{2}{W} e^{-\int_s^t \alpha(u)du} \|\mathbf{z}^*(s) - \mathbf{z}^{**}(s)\|_{1D}, \tag{36}$$

for any initial conditions $s \ge 0$, $\mathbf{p}^*(s)$, $\mathbf{p}^{**}(s)$ *and any* $t \ge s$.

If in addition $D\left(\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\right) \ge \mathbf{0}$, *then*

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1D} \ge e^{-\int_s^t \beta(u)du} \|\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\|_{1D}, \tag{37}$$

for any $0 \le s \le t$.

If the Markov chain is homogeneous, then all elements $b^*_{ij}(t)$ of the matrix $DB(t)D^{-1}$ do not dependent on t i.e. the quantities in (21) are constants. Thus instead of general bounds given by Theorem 1, one can specify then and obtain the following theorem.

Theorem 2. *Let there exist an increasing sequence* $\{d_j,\ j \ge 1\}$ *of positive numbers with* $d_1 = 1$, *such that (19) holds, and* $\alpha(t) = \alpha$ *defined by (21) is positive i.e.* $\alpha > 0$. *Then the Markov chain* $\{X(t),\ t \ge 0\}$ *is strongly ergodic and the following bounds hold:*

$$\begin{aligned} e^{-\chi t} \|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\|_{1D} & \le \|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1D} \\ & \le e^{-\alpha t} \|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\|_{1D}, \end{aligned} \tag{38}$$

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \le 4e^{-\alpha t} \|\mathbf{z}^*(0) - \mathbf{z}^{**}(0)\|_{1D}, \tag{39}$$

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1E} \le \frac{2}{W} e^{-\alpha t} \|\mathbf{z}^*(0) - \mathbf{z}^{**}(0)\|_{1D}, \tag{40}$$

for any initial conditions $s \ge 0$, $\mathbf{p}^*(0)$, $\mathbf{p}^{**}(0)$ *and any* $t \ge 0$.

If in addition $D\left(\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\right) \ge \mathbf{0}$, *then*

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1D} \ge e^{-\beta t} \|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\|_{1D}, \tag{41}$$

for any $t \ge 0$.

For the decay parameter α^* *defined as*

$$\lim_{t\to\infty}(p_{ij}(t)-\pi_j)=O(e^{-\alpha^* t}),$$

where $\{\pi_j,\ j\geq 0\}$ *are the stationary probabilities of the chain, it holds that* $\alpha^*\geq\alpha$.

Notice that some additional results related to *Theorem* 2 can also be found in [4, 6]. If one assumes that the intensities $q_{ij}(t)$ are $1-$periodic in t i.e. $q_{ij}(t)$ are periodic functions and the length of the period is equal to one, then the Markov chain $\{X(t),\ t\geq 0\}$ has the limiting $1-$periodic limiting regime. Under the assumptions of Theorem 1 the Markov chain $\{X(t),\ t\geq 0\}$ is exponentially weakly ergodic. The detailed discussion of this results is given in [27].

Consider now a bit more detailed analysis of two special cases: homogeneous case and the case with periodic intensities. Firstly note that in the both cases there exist positive M and a such that

$$e^{-\int_s^t\alpha(u)\,du}\leq Me^{-a(t-s)} \tag{42}$$

for any $0\leq s\leq t$. Hence the Markov chain $\{X(t),\ t\geq 0\}$ is *exponentially* weakly ergodic. Indeed, if the Markov chain $\{X(t),\ t\geq 0\}$ is homogeneous, then one may put $M=1$, $a=\alpha$ given by (21). If all the intensity functions $q_{ij}(t)$ are $1-$periodic in t, then one may put

$$a=\int_0^1\alpha(t)\,dt,\quad M=e^K,\quad K=\sup_{|t-s|\leq 1}\int_s^t\alpha(u)\,du.$$

By doing so, for any solution of (5) the following bound holds:

$$\begin{aligned}\|\mathbf{z}(t)\|_{1D}\\ \leq\ \|V(t)\|_{1D}\|\mathbf{z}(0)\|_{1D}+\int_0^t\|V(t,\tau)\|_{1D}\|\mathbf{f}(\tau)\|_{1D}\,d\tau\\ \leq\ Me^{-at}\|\mathbf{z}(0)\|_{1D}+\frac{FM}{a},\end{aligned} \tag{43}$$

where F is such that $\|\mathbf{f}(t)\|_{1D}\leq F$ for almost all $t\in[0,1]$. Hence one has the upper bound for the limit

$$\limsup_{t\to\infty}\|\mathbf{z}(t)\|_{1D}\leq\frac{FM}{a}, \tag{44}$$

for any initial condition and

$$\|\mathbf{p}(0)-\mathbf{e}_0\|_{1D}=\|\mathbf{p}(0)\|_{1D}=\|\mathbf{z}(0)\|_{1D}\leq\limsup_{t\to\infty}\|\mathbf{z}(t)\|_{1D}, \tag{45}$$

where $\mathbf{e}_i$ denotes the unit vector of zeros with 1 in the i-th place. If the initial distribution is $\mathbf{p}^{**}(0) = \mathbf{e}_0$ then $\mathbf{z}^{**}(0) = \mathbf{0}$, $\mathbf{z}(t) \geq 0$ for any $\mathbf{p}^*(0)$ and any $t \geq 0$. Therefore

$$\begin{aligned} \|\mathbf{z}(t)\|_{1D} &= d_1 p_1 + (d_1 + d_2) p_2 \\ &\quad + (d_1 + d_2 + d_3) p_3 + ... \\ = d_1 p_1 + \frac{d_1 + d_2}{2} 2 p_2 &+ \frac{d_1 + d_2 + d_3}{3} 3 p_3 + ... \\ &\geq \inf_k \frac{d_1 + ... + d_k}{k} \|\mathbf{z}(t)\|_{1E}, \end{aligned}$$

and one can use $W^* = \inf_k \frac{d_1 + ... + d_k}{k}$ instead of $W = \inf_k \frac{d_k}{k}$, given by (8) in all the bounds on the rate of convergence. Finally, for the considered two special cases one has the following two corollaries.

Corollary 1. *Let $\{X(t),\ t \geq 0\}$ be a homogeneous Markov chain and let there exist an increasing sequence $\{d_j,\ j \geq 1\}$ of positive numbers with $d_1 = 1$ such that (19) holds and in addition $\alpha > 0$. Then the Markov chain $\{X(t),\ t \geq 0\}$ is exponentially ergodic and the following bounds hold:*

$$\|\pi - \mathbf{p}(t, 0)\| \leq \frac{4F}{\alpha} e^{-\alpha t}, \tag{46}$$

$$|\varphi - E(t, 0)| \leq \frac{F}{\alpha W^*} e^{-\alpha t}, \tag{47}$$

where $\pi = (\pi_0, \pi_1, \ldots)^T$ denotes the vector of stationary probabilities of the chain and $\varphi = \sum_{j=0}^{\infty} j \pi_j$ and $\mathbf{p}(0, 0) = \mathbf{e}_0$.

Corollary 2. *Assume that all the intensity functions of the Markov chain $\{X(t),\ t \geq 0\}$ are $1-$periodic in t. Let there exist an increasing sequence $\{d_j,\ j \geq 1\}$ of positive numbers with $d_1 = 1$ such that (19) holds and in addition $\int_0^1 \alpha(t)\, dt = a > 0$. Then the Markov chain $\{X(t),\ t \geq 0\}$ is exponentially weakly ergodic and the following bounds hold:*

$$\|\pi(t) - \mathbf{p}(t, 0)\| \leq \frac{4FM}{a} e^{-at}, \tag{48}$$

$$|\varphi(t) - E(t, 0)| \leq \frac{FM}{a W^*} e^{-at}, \tag{49}$$

where $\pi(t) = (\pi_0(t), \pi_1(t), \ldots)^T$ denotes the vector of limiting probabilities of the chain and $\varphi(t) = \sum_{j=0}^{\infty} j \pi_j(t)$ and $\mathbf{p}(0, 0) = \mathbf{e}_0$.

If the state space of the Markov chain is finite there exist a number of special results (see [4, 6, 29]).

Perturbation Bounds

Let $\{\bar{X}(t), t \geq 0\}$ be a perturbed Markov chain with transposed intensity matrix $\bar{A}(t)$ and the same restrictions as imposed on $A(t)$. It is assumed that $\|\hat{A}(t)\| = \|A(t) - \bar{A}(t)\| \leq \varepsilon$, for almost all $t \geq 0$, which means the perturbations are considered to be small in l_1 norm.

We can obtain the corresponding perturbation bounds. There are two different approaches.

The first approach in this direction are given in [8, 23] both for the discrete and continuous time Markov chains respectively. In the considered situation of Markov chains with continuous time, this approach is based on a comparison of the Cauchy operators of two linear equations in a Banach space considered in [2]. Consider Eq. (5) for the perturbed chain:

$$\frac{d}{dt}\bar{\mathbf{z}}(t) = \bar{B}(t)\bar{\mathbf{z}}(t) + \bar{\mathbf{f}}(t). \tag{50}$$

In this case, the weight space l_{1D} is considered as the base one, and the norms of perturbations are assumed to be small both in l_1 and l_{1D} norms. Namely, we suppose that $\|\hat{B}(t)\|_{1D} = \|B(t) - \bar{B}(t)\|_{1D} \leq \varepsilon$, and $\|\mathbf{f}(t) - \bar{\mathbf{f}}(t)\|_{1D} \leq \varepsilon$, for almost all $t \geq 0$.

The corresponding general results have been obtained in [32]. A typical statement of this kind is as follows:

Theorem 3. *Let the assumptions of Theorem 1 be fulfilled, and let, an addition, $X(t)$ be exponentially weakly ergodic in l_{1D} norm with the corresponding parameters M_D, a_D in (42). Then the following perturbation bound holds:*

$$\limsup_{t\to\infty} \|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\|_1 \leq \frac{4M_D\varepsilon\,(M_D F + a_D)}{a_D\,(a_D - M_D\varepsilon)}, \tag{51}$$

where $\|\bar{\mathbf{f}}(t)\|_{1D} \leq F$ for almost all $t \geq 0$.

The second approach also began with [23], namely, Mitrophanov [14] successfully applied probabilistic considerations and ergodicity in uniform operator topology which allowed to significantly reduce the constant factor in the stability estimate. The corresponding bounds for inhomogeneous situation has been obtained in [28].

A typical statement of this kind is as follows:

Theorem 4. *Let Markov cain $X(t)$ be exponentially weakly ergodic in l_1 norm with the corresponding parameters M^*, α^* in (42). Then the following bound holds:*

$$\limsup_{t\to\infty} \|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\|_1 \leq \frac{\varepsilon\,(1 + \log M^*)}{\alpha^*}. \tag{52}$$

Truncation Bounds

Calculation of the limiting characteristics for (inhomogeneous) birth-death processes via truncations was firstly mentioned in [24] and was considered in details in [27]. First results for more general Markovian queueing models have been obtained recently in [31]. The respective bound of approximation error as a rule depends on time. Vladimir V. Kalashnikov in the early 1990-s suggested that in some cases one can obtain uniform (in time) error bounds of truncation. Such bounds for inhomogeneous birth-death processes have been obtained in [30], and for more general Markov chains in [33], this statement can be formulated in the following way.

Let $X_{N-1}(t)$ be a truncated process with the state space $\mathrm{E}_{N-1} = \{0, 1, \ldots, N-1\}$ and the corresponding transposed infinitesimal matrix

$$A_{N-1}(t) = \begin{pmatrix} b_{00}(t) & a_{01}(t) & \cdots & a_{0,N-1}(t) \\ a_{10}(t) & b_{11}(t) & \cdots & a_{1,N-1}(t) \\ a_{20}(t) & a_{21}(t) & \cdots & a_{2,N-1}(t) \\ \cdots & \cdots & \cdots & \cdots \\ a_{N-1,0}(t) & a_{N-1,1}(t) & \cdots & b_{N-1,N-1}(t) \end{pmatrix},$$

where $b_{ii}(t) = -\sum_{k=0, k\neq i}^{N-1} a_{ki}(t)$.

Then the forward Kolmogorov system for $X_{N-1}(t)$ is

$$\frac{d\mathbf{p}^*}{dt} = A_{N-1}(t)\mathbf{p}^*,$$

and instead of (5) we have

$$\frac{d\mathbf{z}^*}{dt} = B_{N-1}(t)\mathbf{z}^*(t) + \mathbf{f}_{N-1}(t), \tag{53}$$

where $\mathbf{f}_{N-1}(t) = (a_{10}(t), a_{20}(t), \ldots, a_{N-1,0}(t))^\top$, $\mathbf{z}^*(t) = (p_1, p_2, \cdots, p_{N-1})^\top$,

$$B_{N-1} = \begin{pmatrix} b_{11}(t) - a_{10}(t) & a_{12}(t) - a_{10}(t) & \cdots & a_{1,N-1}(t) - a_{10}(t) \\ a_{21}(t) - a_{20}(t) & b_{22}(t) - a_{20}(t) & \cdots & a_{2,N-1}(t) - a_{20}(t) \\ \cdots & \cdots & \cdots & \cdots \\ a_{N-1,1}(t) - a_{N-1,0}(t) & a_{N-1,2}(t) - a_{N-1,0}(t) & \cdots & b_{N-1,N-1}(t) - a_{N-1,0}(t) \end{pmatrix}.$$

Below we will identify the finite vector with entries $(a_1, \ldots, a_{N-1})^\top$ and the infinite vector with the same first $N-1$ coordinates and the others equal to zero. Moreover we suppose that

$$a_{i+k,i}(t) = q_{i,i+k}(t) \le R \cdot q^{-k}, \qquad q > 1, \quad R > 0, \tag{54}$$

for any $k \ge 1$, $i \ge 0$ and almost all $t \ge 0$. For $\delta \in (1, \sqrt{q})$ we consider the sequences $d_k = \delta^{k-1}$ and $d_k^* = \delta^{2k-2}$, $k \ge 1$.

Denote

$$W = \inf_{i \ge 1} \frac{d_i}{i}, \qquad g_i = \sum_{n=1}^{i} d_n.$$

Let D and D^* be upper triangular matrices:

$$D = \begin{pmatrix} d_1 & d_1 & d_1 & \cdots \\ 0 & d_2 & d_2 & \cdots \\ 0 & 0 & d_3 & \cdots \\ & \ddots & \ddots & \ddots \end{pmatrix}, \quad D^* = \begin{pmatrix} d_1^* & d_1^* & d_1^* & \cdots \\ 0 & d_2^* & d_2^* & \cdots \\ 0 & 0 & d_3^* & \cdots \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

and l_{1D}, l_{1D^*} be the corresponding spaces of sequences:

$$l_{1D} = \{\mathbf{z} = (p_1, p_2, \ldots)^\top \mid \|\mathbf{z}\|_{1D} \equiv \|D\mathbf{z}\|_1 < \infty\},$$
$$l_{1D^*} = \{\mathbf{z} = (p_1, p_2, \ldots)^\top \mid \|\mathbf{z}\|_{1D^*} \equiv \|D^*\mathbf{z}\|_1 < \infty\}.$$

We suppose that there exist positive constants M, a, M^*, a^* such that the following bounds

$$\|V(t,s)\|_{1D} \le Me^{-a(t-s)}, \tag{55}$$

and

$$\|V(t,s)\|_{1D^*} \le M^* e^{-a^*(t-s)}, \tag{56}$$

hold for Cauchy operator $V(t,s)$ of Eq. (5) for any s, t $(0 \le s \le t)$. These estimates guarantee exponential convergence to zero as $t - s \to \infty$ of the difference $\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \to 0$ in l_{1D} and l_{1D^*} norms respectively for the corresponding initial conditions.

Theorem 5. *Let the assumptions* (54), (55), (56) be fulfilled. *Then the following bounds of error of truncations hold:*

$$\|\mathbf{p}(t) - \mathbf{p}_{N-1}(t)\| \le C_1\Big(\frac{\delta^2}{q}\Big)^{N/3} + C_2\delta^{-N/3} + C_3\Big(\frac{\delta}{q}\Big)^{N} \tag{57}$$

and

$$|E(t,0) - E_{N-1}(t,0)| \le \frac{1}{W}\Big\{C_1\Big(\frac{\delta^2}{q}\Big)^{N/3} + C_2\delta^{-N/3} + C_3\Big(\frac{\delta}{q}\Big)^{N}\Big\}, \tag{58}$$

where index $N-1$ shows the corresponding characteristics of truncated process and $X(0) = X_{N-1}(0) = 0$. Moreover, constants $C_i = C_i(\delta, q)$ do not depend on N and on $\delta \in (1, \sqrt{q})$.

Finally we can briefly describe the possible procedure for finding $\pi(t)$ and $\varphi(t)$ in case of 1−periodic in t intensities. Firstly we estimate the instant $t = t^*$ (using the ergodicity bounds), starting from which the solution of the forward Kolmogorov system (3) with the initial condition $X(0)$ is within the fixed $\epsilon > 0$ from the limiting periodic probabilities. Then we estimate the size n^* of the state space $\{0, 1, \dots, n^*\}$, which guarantees the desired approximation error on the interval $[0, t^* + 1]$. Then we find the solution of the truncated system on the interval $[0, t^* + 1]$, eventually the values for $\pi(t)$ and $\varphi(t)$ on the interval $[t^*, t^* + 1]$.

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On the study of forward Kolmogorov system and the corresponding problems for inhomogeneous continuous-time Markov chains

Alexander Zeifman

Abstract An inhomogeneous continuous-time Markov chain $X(t)$ with finite or countable state space under some natural additional assumptions is considered. As a consequence, we study a number of problems for the corresponding forward Kolmogorov system, which is the linear system of differential equations with special structure of the matrix $A(t)$. In the countable situation we have an equation in the space of sequences l_1. The important properties of $X(t)$ (such as weak and strong ergodicity, perturbation bounds, truncation bounds) are closely connected with behaviour of the solutions of the forward Kolmogorov system as $t \to \infty$. The main problems and some approaches for their solution are discussed in the paper.

1 Introduction

Continuous-time Markov chains are widely used for the study of stochastic models in the natural and technical sciences, such as queuing theory, biology, chemistry, etc.

Let $\{X(t),\ t \geq 0\}$ be a continuous-time Markov chain with state space $\mathscr{X} = \{0,1,2\ldots\}$. Denote by $p_{ij}(s,t) = P\{X(t) = j\,|X(s) = i\}$, $i,j \geq 0$, $0 \leq s \leq t$ the transition probabilities of $X(t)$ and by $p_i(t) = P\{X(t) = i\}$ – the probability that the Markov chain $X(t)$ is in state i at time t. Let $\mathbf{p}(t) = (p_0(t), p_1(t), \ldots)^T$ be the probability distribution vector at instant t. Throughout the paper we assume that in an element of time h the possible transitions and their associated probabilities are

$$p_{ij}(t,t+h) = \begin{cases} q_{ij}(t)h + \alpha_{ij}(t,h), & \text{if } j \neq i \\ 1 + q_{ii}(t)h + \alpha_i(t,h), & \text{if } j = i, \end{cases} \quad i,j \in \mathscr{X}, \tag{1}$$

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where all the $\alpha_i(t,h)$ are $o(h)$ uniformly in i, i.e. $\sup_i |\alpha_i(t,h)| = o(h)$ and

$$q_{ii}(t) = -\sum_{k\in\mathscr{X},k\neq i} q_{ik}(t).$$

The matrix $Q(t) = (q_{ij}(t))_{i,j=0}^{\infty}$ is called the intensity (or infinitesimal) matrix of the chain $\{X(t),\ t\geq 0\}$.

The Markov chain $X(t)$ is called homogeneous if Q is a constant matrix, and it is called inhomogeneous in the opposite case.

As a rule, in the inhomogeneous case we will assume that the intensity functions $q_{ij}(t)$ are locally integrable on the interval $[0,\infty)$.

Henceforth it is assumed that the $Q(t)$ is essentially bounded, i.e.

$$\sup_i |q_{ii}(t)| = L(t) \leq L < \infty, \tag{2}$$

for almost all $t\geq 0$.

In many problems, condition (2) can be weakened and replaced by $L(t) < \infty$, for almost all $t\geq 0$.

Then the probabilistic dynamics of the process $\{X(t),\ t\geq 0\}$ is given by the forward Kolmogorov system

$$\frac{d}{dt}\mathbf{p}(t) = A(t)\mathbf{p}(t), \tag{3}$$

where $A(t) = Q^T(t)$ is the transposed intensity matrix. All column sums of this matrix are zeros for any $t\geq 0$, and $A(t)$ is essentially nonnegative (i.e. all its off-diagonal elements are nonnegative for any $t\geq 0$).

Throughout the paper by $\|\cdot\|$ we denote the l_1-norm, i. e. $\|\mathbf{p}(t)\| = \sum_{k\in\mathscr{X}} |p_k(t)|$, and $\|Q(t)\| = \sup_{j\in\mathscr{X}} \sum_{i\in\mathscr{X}} |q_{ij}|$. Let Σ be a set all stochastic vectors, i. e. l_1 vectors with non-negative coordinates and unit norm. Hence we have $\|A(t)\| = 2\sup_{k\in\mathscr{X}} |q_{kk}(t)| \leq 2L$ for almost all $t\geq 0$. Hence the operator function $A(t)$ from l_1 into itself is bounded for almost all $t\geq 0$ and locally integrable on $[0;\infty)$. Therefore we can consider (3) as a differential equation in the space l_1 with bounded operator.

It is well known (see [2]) that the Cauchy problem for differential equation (3) has a unique solutions for an arbitrary initial condition, and $\mathbf{p}(s)\in\Sigma$ implies $\mathbf{p}(t)\in\Sigma$ for $t\geq s\geq 0$.

Denote by $E(t,k) = E(X(t)|X(0)=k)$ the conditional expected number of 'particles' in the system at instant t, provided that initially (at instant $t=0$) k 'particles were present in the system.

In order to obtain perturbation bounds we consider a class of perturbed Markov chains $\{\bar{X}(t), t\geq 0\}$ defined on the same state space $\mathscr{X}$ as the original Markov chain $\{X(t), t\geq 0\}$, with the intensity matrix $\bar{A}(t)$ and the same restrictions as imposed on $A(t)$. It is assumed that $\|\hat{A}(t)\| = \|A(t)-\bar{A}(t)\| \leq \varepsilon$, for almost all $t\geq 0$, which means the perturbations are considered to be small.

Before proceeding to the derivation of the main results of the paper, we recall two definitions. Recall that a Markov chain $\{X(t),\ t\geq 0\}$ is called *weakly ergodic*, if $\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \to 0$ as $t \to \infty$ for any initial conditions $\mathbf{p}^*(0)$ and $\mathbf{p}^{**}(0)$, where $\mathbf{p}^*(t)$ and $\mathbf{p}^{**}(t)$ are the corresponding solutions of (3). A Markov chain $\{X(t),\ t \geq 0\}$ has the limiting mean $\varphi(t)$, if $\lim_{t\to\infty}(\varphi(t) - E(t,k)) = 0$ for any k.

It is clear, therefore, that the study of the qualitative properties and the derivation of estimates for Markov chains with continuous time is reduced to the study of the corresponding properties of solutions of the forward Kolmogorov (3) system on Σ.

A general approach to obtaining sharp bounds on the rate of convergence via the notion of the logarithmic norm of an operator function was recently discussed in detail in our papers [34, 35, 36]. The first studies in this direction were published since 1980-s for birth-death models, see [25, 26]. In [34, 35] we have highlighted four fairly broad classes of finite and countable Markov chains, for which the forward Kolmogorov system can be transformed into a system with an essentially nonnegative matrix. Moreover, it turns out that similar results can be obtained for some other models, see, for example [37]. Computation of the limiting characteristics for such chains using bounds on the rate of convergence and truncations technique introduced in [30, 33].

The approach is based on studying the norm of the Cauchy operator of the reduced forward Kolmogorov system by estimation of the so-called logarithmic norm of an operator function. The method of the complete study of the process $X(t)$ that describes the number of claims in the system assumes the construction of a) upper bounds for the rate of convergence of the limit mode, providing that, beginning from a certain time, say, t^*, the probability characteristics of the process $X(t)$ do not depend on the initial conditions (up to a given discrepancy); b) analogous lower bounds which are also very important and provide that the "independence" of the initial conditions cannot appear before a certain time, say, t_*; c) stability bounds providing that if the structure of the matrix of intensities of the process is taken into account in an appropriate way, and the errors in intensities are small, then the basic characteristics of the process are calculated in an adequate way; d) approximations to the process by means of truncation by similar processes with a lesser number of states and construction of the corresponding estimates for the error. Finally, applying the results of a), c), d) to the system with 1-time-periodic intensities and solving the forward Kolmogorov system with the simplest initial condition $X(0) = 0$ for the truncated process on the interval $[t^*, t^* + 1]$, as a result we obtain all basic probability characteristics of both the process $X(t)$, and close "perturbed" processes. Note that the item a) is most important, because after the corresponding bounds are obtained, the solutions of other problems can be constructed automatically on the base of the results of [27]-[34].

Generally speaking, instead of obtaining the solution to the Cauchy problem on a short time interval by some methods that are approximate anyway, which does not provide actual information of the real basic properties of the system, we determine the time interval, on which the Cauchy problem for the forward Kolmogorov system must really be solved and find this solution.

It is worth noting that exact estimates of the rate of convergence yield exact estimates of stability (perturbation bounds), see [8, 11, 14, 15, 17, 23, 32] and references therein. Moreover, such connections and their significance were highlighted in the recent communication by Mitrophanov, see

http://alexmitr.com/talk_DDE2018_Mitrophanov_FIN_post_sm.pdf.

The approach is based on the special properties of linear systems of differential equations with essentially nonnegative matrices. Specifically, if the column-wise sums of the elements of this matrix are identical and equal to, say, $-\alpha^*(t)$, then the exact upper bound of order $\exp\left\{-\int_0^t \alpha^*(u)\,du\right\}$ can be obtained for the rate of convergence of the solutions of the system in the corresponding metric. Moreover, if the column-wise sums of the absolute values of the elements of this matrix are identical and equal to, say, $\chi^*(t)$, then the exact lower bound of order $\exp\left\{-\int_0^t \chi^*(u)\,du\right\}$ can be obtained for the convergence rate as well. The bounds are obtained in three steps. At first step one excludes the (0) state from the forward Kolmogorov system of differential equations and thus obtains the new system with the new intensity matrix which is, in general, not non-diagonally non-negative. The second step is to transform the new intensity matrix in such a way that non-diagonally elements are non-negative and which leads to (loosely speaking) least distance between specifically defined upper and lower bounds. At third step one uses the logarithmic norm for the estimation of the convergence rate.

Here the key step is the second one. The transformation is made using a sequence of positive numbers $\{d_i, i \geq 1\}$, which does not have any probabilistic meaning and can be considered as an analogue of Lyapunov functions.

The advantages of this three-step approach is that it allows one to deal with time-homogeneous and time-inhomogeneous processes and it leads to exact both upper and lower bounds for the convergence rate. In time-homogeneous case the approach allows one to obtain the corresponding bounds for the decay parameter and gives an explicit bounds in total variation norm.

2 General transformations

Recall that one has introduced $A(t)$ as the transposed intensity matrix $Q(t)$. Thus it has the form

$$A(t) = \begin{pmatrix} a_{00}(t) & a_{01}(t) & \cdots & a_{0r}(t) & \cdots \\ a_{10}(t) & a_{11}(t) & \cdots & a_{1r}(t) & \cdots \\ a_{20}(t) & a_{21}(t) & \cdots & a_{2r}(t) & \cdots \\ \cdots \\ a_{r0}(t) & a_{r1}(t) & \cdots & a_{rr}(t) & \cdots \\ \cdots \end{pmatrix}, \tag{4}$$

where $a_{ii}(t) = -\sum_{k\in\mathscr{X}, k\neq i} a_{ki}(t)$. Since $p_0(t) = 1 - \sum_{i=1}^{\infty} p_i(t)$ due to normalization condition, one can rewrite the system (3) as follows:

$$\frac{d}{dt}\mathbf{z}(t) = B(t)\mathbf{z}(t) + \mathbf{f}(t), \tag{5}$$

where

$$\mathbf{f}(t) = (a_{10}(t), a_{20}(t), \dots)^T\,,\ \mathbf{z}(t) = (p_1(t), p_2(t), \dots)^T\,,$$

$$B(t) = \begin{pmatrix} a_{11}-a_{10} & a_{12}-a_{10} & \cdots & a_{1r}-a_{10} & \cdots \\ a_{21}-a_{20} & a_{22}-a_{20} & \cdots & a_{2r}-a_{20} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{r1}-a_{r0} & a_{r2}-a_{r0} & \cdots & a_{rr}-a_{r0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{6}$$

Each entry of B depends on t. See detailed discussion of this transformation in [7, 27].

There is the following simple relationship between pairs, $\mathbf{z}^{(i)} = \mathbf{z}^{(i)}(t),\ t \geq 0,\ i = 1,2$, of solutions of (5) and pairs of solutions of (3), $\mathbf{p}^{(i)} = \mathbf{p}^{(i)}(t),\ t \geq 0,\ i = 1,2$:

$$\begin{aligned} \left\|\mathbf{p}^{(1)} - \mathbf{p}^{(2)}\right\|_1 &= \left|p_0^{(1)} - p_0^{(2)}\right| + \sum_{i\geq 1}\left|p_i^{(1)} - p_i^{(2)}\right| = \\ &= \left|1 - \sum_{i\geq 1} p_i^{(1)} - \left(1 - \sum_{i\geq 1} p_i^{(2)}\right)\right| + \left\|\mathbf{z}^{(1)} - \mathbf{z}^{(2)}\right\|_1 = \\ &= \left|\sum_{i\geq 1}\left(p_i^{(2)} - p_i^{(1)}\right)\right| + \left\|\mathbf{z}^{(1)} - \mathbf{z}^{(2)}\right\|_1 \leq \sum_{i\geq 1}\left|p_i^{(2)} - p_i^{(1)}\right| + \left\|\mathbf{z}^{(1)} - \mathbf{z}^{(2)}\right\|_1 = \\ &= 2\left\|\mathbf{z}^{(1)} - \mathbf{z}^{(2)}\right\|_1, \quad t \geq 0. \end{aligned}$$

Consequently,

$$\left\|\mathbf{z}^{(1)} - \mathbf{z}^{(2)}\right\|_1 \leq \left\|\mathbf{p}^{(1)} - \mathbf{p}^{(2)}\right\|_1 \leq 2\left\|\mathbf{z}^{(1)} - \mathbf{z}^{(2)}\right\|_1, \quad t \geq 0, \tag{7}$$

which will be used in the study of stability and ergodicity.

Let $\{d_i,\ i \geq 1\}$ with $d_1 = 1$ be an increasing sequence of positive numbers. Put

$$W = \inf_{i\geq 1}\frac{d_i}{i}. \tag{8}$$

and denote by D the upper triangular matrix of the following form:

$$D = \begin{pmatrix} d_1 & d_1 & d_1 & \cdots \\ 0 & d_2 & d_2 & \cdots \\ 0 & 0 & d_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{9}$$

Let l_{1D} be the corresponding space of sequences

$$l_{1D} = \left\{ \mathbf{z}(t) = (p_1(t), p_2(t), \cdots)^T \,|\, \|\mathbf{z}(t)\|_{1D} \equiv \|D\mathbf{z}(t)\|_1 < \infty \right\}$$

and introduce also the auxiliary norm $\|\cdot\|_{1E}$ defined as $\|\mathbf{z}(t)\|_{1E} = \sum_{k=1}^{\infty} k|p_k(t)|$. Then in $\|\cdot\|_{1D}$ norm the following two inequalities hold:

$$\begin{aligned}
\|\mathbf{z}(t)\|_{1D} = d_1 \left|\sum_{i=1}^{\infty} p_i(t)\right| + d_2 \left|\sum_{i=2}^{\infty} p_i(t)\right| + \\
+ d_3 \left|\sum_{i=3}^{\infty} p_i(t)\right| + \ldots \geq \\
\geq \left(\left|\sum_{i=1}^{\infty} p_i(t)\right| + \left|\sum_{i=2}^{\infty} p_i(t)\right| + \left|\sum_{i=3}^{\infty} p_i(t)\right| + \ldots \right) \geq \\
\geq \frac{1}{2} \left(\left(\left|\sum_{i=1}^{\infty} p_i(t)\right| + \left|\sum_{i=2}^{\infty} p_i(t)\right| \right) + \right. \\
+ \left(\left|\sum_{i=2}^{\infty} p_i(t)\right| + \left|\sum_{i=3}^{\infty} p_i(t)\right| \right) + \ldots \geq \\
\geq \frac{1}{2} \sum_{i=1}^{\infty} |p_i(t)| = \frac{1}{2} \|\mathbf{z}(t)\|_1,
\end{aligned} \tag{10}$$

$$\begin{aligned}
\|\mathbf{z}(t)\|_{1E} = \sum_{k=1}^{\infty} k|p_k(t)| = \\
= \sum_{k=1}^{\infty} \frac{k}{d_k} d_k |p_k(t)| \leq W^{-1} \sum_{k=1}^{\infty} d_k |p_k(t)| = \\
= W^{-1} \sum_{k=1}^{\infty} d_k \left| \sum_{i=k}^{\infty} p_i(t) - \sum_{i=k-1}^{\infty} p_i(t) \right| \\
\leq W^{-1} \sum_{k=1}^{\infty} d_k \left(\left| \sum_{i=k}^{\infty} p_i(t) \right| + \left| \sum_{i=k-1}^{\infty} p_i(t) \right| \right) \leq \\
\leq \frac{2}{W} \sum_{k=1}^{\infty} d_k \left| \sum_{i=k}^{\infty} p_i(t) \right| \leq \frac{2}{W} \|\mathbf{z}(t)\|_{1D}.
\end{aligned} \tag{11}$$

3 Logarithmic norm and related bounds

Recall here the definition of logarithmic norm.

The concept of *logarithmic norm* of a square matrix was developed independently by Dahlquist [1] and Lozinskiĭ [12] as a tool to derive error bounds in the numerical integration of initial-value problems for a system of ordinary differential equations (see also the survey papers [21] and [20]). For the linear differential equation in a Banach space with locally integrable operator function this notion was discussed in [2].

Let $B(t)$, $t \geq 0$ be a one-parameter family of bounded linear operators on a Banach space $\mathscr{B}$ and let I denote the identity operator.
For each $t \geq 0$, the number

$$\gamma(B(t)) = \lim_{h \to +0} \frac{\|I + hB(t)\| - 1}{h} \tag{12}$$

is called the logarithmic norm of the operator $B(t)$.
The logarithmic norm of the matrix $B(t) = \{b_{ij}(t)\}$, $t \geq 0$ corresponding to a linear operator on the vector space $\mathscr{B}$ equipped with ℓ_1- norm, is

$$\gamma(B(t)) = \sup_j \left(b_{jj}(t) + \sum_{i \neq j} |b_{ij}(t)| \right), \quad t \geq 0. \tag{13}$$

Associate now the family of operators $B(t)$, $t \geq 0$ with the system of differential equations

$$\frac{d\mathbf{x}}{dt} = B(t)\mathbf{x}, \quad t \geq 0, \tag{14}$$

where the functions $b_{ij}(t)$, $0 \leq i, j < \infty$ are assumed to be locally integrable on $[0, \infty)$, and denote by $V(t,s)$, $0 \leq s \leq t$ the corresponding Cauchy operator (hence $\mathbf{x}(t) = V(t,s)\mathbf{x}(s)$ for any $0 \leq s \leq t$). Then the logarithmic norm of the operator $B(t)$ is related to $V(t,s)$, $0 \leq s \leq t$ by

$$\gamma(B(t)) = \lim_{h \to +0} \frac{\|V(t+h,t)\| - 1}{h}, \quad t \geq 0. \tag{15}$$

From the latter one can deduce the following bounds on the $\mathscr{B}$-norm of the Cauchy operator $V(t,s)$, $0 \leq s \leq t$:

$$e^{-\int_s^t \gamma(-B(\tau))\, d\tau} \leq \|V(t,s)\| \leq e^{\int_s^t \gamma(B(\tau))\, d\tau}, \quad 0 \leq s \leq t. \tag{16}$$

Moreover, for any solution $\mathbf{x}(t) \in \mathscr{B}$, $t \geq 0$ of (14) we have

$$\|\mathbf{x}(t)\| \geq e^{-\int_s^t \gamma(-B(\tau))\, d\tau} \|\mathbf{x}(s)\|. \tag{17}$$

We will also make use of the fact that if $\mathscr{B}$ is a vector space with norm ℓ_1 and all diagonal elements of B are non-negative then, by (13)

$$\gamma(B(t)) = \sup_j \sum_i b_{ij}(t), \quad t \geq 0,$$

and, *a fortiori*, for any solution $\mathbf{x}(t)$, $t \geq 0$ of (14), s.t. $\mathbf{x}(s) \geq \mathbf{0}$, we have

$$\|\mathbf{x}(t)\| \geq e^{\int_s^t \inf_j \sum_i b_{ij}(\tau)\, d\tau} \|\mathbf{x}(s)\|, \quad 0 \leq s \leq t. \tag{18}$$

Consider the equation (5) in the space l_{1D}, where $B(t)$ and $\mathbf{f}(t)$ are locally integrable on $[0,+\infty)$. Let one compute the logarithmic norm of operator function $B(t)$.

Then for the logarithmic norm of the operator function $B(t)$ in $\|\cdot\|_{1D}$ norm the following equality holds:

$$\gamma(B(t))_{1D} = \gamma(DB(t)D^{-1})_1.$$

Denote by $B^*(t) = DB(t)D^{-1}$, and the elements of $B^*(t)$ by $b^*_{ij}(t)$ i.e. $B^*(t) = \left(b^*_{ij}(t)\right)_{i,j=1}^{\infty}$. Assume that

$$b^*_{ij}(t) \geq 0,\ i \neq j,\ t \geq 0. \tag{19}$$

Remark 1. Note that assumption (19) of essential nonnegativity of the reduced matrix $B^*(t)$ is key to the possibility of effective use of the method of the logarithmic norm. In particular, this assumption is fulfilled for four important classes of Markov chains, which we consider in the next section.

Put

$$\alpha_i(t) = -\sum_{j=0}^{\infty} b^*_{ji}(t), \quad \chi_i(t) = -\sum_{j=0}^{\infty} |b^*_{ji}(t)|,\ i \geq 1, \tag{20}$$

and let $\alpha(t)$ and $\beta(t)$ denote the least lower and the least upper bound of the sequence of functions $\{\alpha_i(t),\ i \geq 1\}$ and $\chi(t)$ denote the least upper bound of $\{\chi_i(t),\ i \geq 1\}$ i.e.

$$\alpha(t) = \inf_{i\geq 1} \alpha_i(t),\ \beta(t) = \sup_{i\geq 1} \alpha_i(t), \tag{21}$$

$$\chi(t) = \sup_{i\geq 1} \chi_i(t). \tag{22}$$

Then the logarithmic norms of $B(t)$ and $(-B(t))$ are equal to

$$\gamma(B(t))_{1D} = \sup_i \alpha_i(t) = -\alpha(t),$$
$$\gamma(-B(t))_{1D} = \sup \chi_i(t) = \chi(t).$$

If now one defines $\mathbf{v}(t) = D(\mathbf{p}^*(t) - \mathbf{p}^{**}(t))$, then the following equation holds

$$\frac{d}{dt}\mathbf{v}(t) = DB(t)D^{-1}\mathbf{v}(t), \tag{23}$$

Notice that due to (19), the inequality $\mathbf{v}(s) \geq \mathbf{0}$ implies that $\mathbf{v}(t) \geq \mathbf{0}$ for any $t \geq s$. Hence

$$\frac{d}{dt}\sum_{i=1}^{\infty} v_i(t) \geq -\beta(t)\sum_{i=1}^{\infty} v_i(t), \tag{24}$$

and one can obtain establish the corresponding bounds on the rate of convergence, perturbation bounds, and estimates on the error of truncations.

4 Four classes of Markov chains

These classes were previously studied in [34, 35]. We use here the terminology from Markov chain theory and queueing in parallel depending on context.

Class (I). Inhomogeneous birth-death processes (BDP), where all $a_{ij}(t) = 0$ for any $t \geq 0$ if $|i-j| > 1$, and $a_{i,i+1}(t) = \mu_{i+1}(t),\ \ a_{i+1,i}(t) = \lambda_i(t)$ - birth and death rates respectively. This process, in particular, is a standard model as queue-length process for a general Markovian queue $M_n(t)/M_n(t)/1$.

In this situation we obtain

$$B^*(t) = \begin{pmatrix} -(\lambda_0(t)+\mu_1(t)) & \mu_1(t) & 0 & \cdots & 0 & \cdots & \cdots \\ \lambda_1(t) & -(\lambda_1(t)+\mu_2(t)) & \mu_2(t) & \cdots & 0 & \cdots & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \\ 0 & \cdots & \cdots & \lambda_{r-1}(t) & -(\lambda_{r-1}(t)+\mu_r(t)) & \mu_r(t) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \tag{25}$$

if $S = \infty$, and

$$B^*(t) = \begin{pmatrix} -(\lambda_0(t)+\mu_1(t)) & \mu_1(t) & 0 & \cdots & 0 \\ \lambda_1(t) & -(\lambda_1(t)+\mu_2(t)) & \mu_2(t) & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & \lambda_{S-1}(t) & -(\lambda_{S-1}(t)+\mu_S(t)) \end{pmatrix}, \tag{26}$$

if $S < \infty$.

One can see that the transformed matrix $B^*(t)$ is essentially nonnegative for any t, that is all off-diagonal elements of this matrix are nonnegative for any t.

Remark 2. This class is the most studied. It includes, in particular, models of systems of the theory of queues $M_t/M_t/N$, and $M_t/M_t/N/N$, see for instance [30, 27, 26, 25, 7, 13, 4, 3, 5, 22] and references therein. For the first one, we get the matrix (25) with $\lambda_k(t) = \lambda(t)$ and $\mu_k(t) = \min(k,N) \cdot \mu(t)$, and for the second one we get (26) with $\lambda_k(t) = \lambda(t)$ and $\mu_k(t) = k\mu(t)$.

Another approach to the study of close models with discrete time was considered in [9].

Class (II). Inhomogeneous queue-length process for a queue with batch arrivals and single services, where $a_{ij}(t) = 0$ for any $t \geq 0$ if $i < j-1$, all arrival rates do not depend on the size of a queue, where $a_{i+k,i}(t) = a_k(t)$ for $k \geq 1$ - the rate of arrival of a group of k customers, $a_{i,i+1}(t) = \mu_{i+1}(t)$ - the service rate. Such models in simplest situations were firstly considered in [16].

In this situation we have

$$B^*(t) = \begin{pmatrix} a_{11}(t) & \mu_1(t) & 0 & \cdots & 0 \\ a_1(t) & a_{22}(t) & \mu_2(t) & \cdots & 0 \\ a_2(t) & a_1(t) & a_{33}(t) & \mu_3(t) & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{27}$$

if $S=\infty$, and

$$B^*(t)=\begin{pmatrix} a_{11}(t)-a_S(t) & \mu_1(t) & 0 & \cdots & 0\\ a_1(t)-a_S(t) & a_{22}(t)-a_{S-1}(t) & \mu_2(t) & \cdots & 0\\ \ddots & \ddots & \ddots & \ddots & \ddots\\ a_{S-1}(t)-a_S(t) & \cdots & \cdots & a_1(t)-a_2(t) & a_{SS}(t)-a_1(t) \end{pmatrix}, \quad (28)$$

if $S<\infty$.

One can see that the transformed matrix $B^*(t)$ is certainly essentially nonnegative for any t if arrival rates $a_k(t)$ are decrease in k.

Class (III). Inhomogeneous queue-length process for the queueing model with batch services and single arrivals, where all $a_{ij}(t)=0$ for any $t\geq 0$ if $i>j+1$, and all service rates do not depend on the size of a queue, where $a_{i,i+k}(t)=b_k(t),\ k\geq 1$ is the rate of service of a group of k customers, and $a_{i+1,i}(t)=\lambda_i(t)$ is the arrival rate, see also [16]. One can find more modern studies of these models in [10].

Here we obtain

$$B^*(t)=\begin{pmatrix} -(\lambda_0(t)+b_1(t)) & b_1(t)-b_2(t) & b_2(t)-b_3(t) & \cdots & \cdots & \\ \lambda_1(t) & -\big(\lambda_1(t)+\sum_{i\leq 2} b_i(t)\big) & b_1(t)-b_3(t) & \cdots & \cdots & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \cdots & \cdots & \lambda_{r-1}(t) & -\big(\lambda_{r-1}(t)+\sum_{i\leq r} b_i(t)\big)\cdots & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \quad (29)$$

if $S=\infty$, and

$$B^*(t)=\begin{pmatrix} -(\lambda_0(t)+b_1(t)) & b_1(t)-b_2(t) & b_2(t)-b_3(t) & \cdots & b_{S-1}(t)-b_S(t)\\ \lambda_1(t) & -\big(\lambda_1(t)+\sum_{i\leq 2} b_i(t)\big) & b_1(t)-b_3(t) & \cdots & b_{S-2}(t)-b_S(t)\\ \ddots & \ddots & \ddots & \ddots & \ddots\\ 0 & \cdots & \cdots & \lambda_{S-1}(t) & -\big(\lambda_{S-1}(t)+\sum_{i\leq S} b_i(t)\big) \end{pmatrix}, \quad (30)$$

if $S<\infty$.

One can see that the transformed matrix $B^*(t)$ is certainly essentially nonnegative for any t if service rates $b_k(t)$ are decrease in k.

Class (IY). Queue-length process for a non-stationary queueing model with batch arrivals and group services, where all rates do not depend on the size of a queue, here $a_{i+k,i}(t)=a_k(t)$, and $a_{i,i+k}(t)=b_k(t)$ for $k\geq 1$ are the rates of arrival and service of a group of k customers respectively. Such process were studied in [18, 19, 31].

$$B^* = \begin{pmatrix} a_{11}(t) & b_1(t)-b_2(t) & b_2(t)-b_3(t) & \cdots & \cdots \\ a_1(t) & a_{22}(t) & b_1(t)-b_3(t) & \cdots & \cdots \\ & & & & \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ a_{r-1}(t) & \cdots & \cdots & a_1(t) & a_{rr}(t) \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \ \cdots \end{pmatrix}, \tag{31}$$

if $S = \infty$, and

$$B^*(t) = \begin{pmatrix} a_{11}(t)-a_S(t) & b_1(t)-b_2(t) & b_2(t)-b_3(t) & \cdots & b_{S-1}(t)-b_S(t) \\ a_1(t)-a_S(t) & a_{22}(t)-a_{S-1}(t) & b_1(t)-b_3(t) & \cdots & b_{S-2}(t)-b_S(t) \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ a_{S-1}(t)-a_S(t) & \cdots & \cdots & a_1(t)-a_2(t) & a_{SS}(t)-a_1(t) \end{pmatrix}, \tag{32}$$

if $S < \infty$.

In this case the transformed matrix $B^*(t)$ is surely essentially nonnegative for any t if all arrival and service rates $a_k(t)$ and $b_k(t)$ are decreasing on k.

5 General bounds for continuous-time Markov chains

RATE OF CONVERGENCE.

Theorem 1. *Let there exist an increasing sequence $\{d_j,\ j \geq 1\}$ of positive numbers with $d_1 = 1$, such that (19) holds, and $\alpha(t)$ defined by (21) satisfies*

$$\int_0^\infty \alpha(t)\, dt = +\infty. \tag{33}$$

Then the Markov chain $\{X(t),\ t \geq 0\}$ is weakly ergodic and the following bounds hold:

$$\begin{aligned} & e^{-\int_s^t \chi(u)du} \|\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\|_{1D} \leq \\ & \leq \|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1D} \leq \\ & \leq e^{-\int_s^t \alpha(u)du} \|\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\|_{1D}, \end{aligned} \tag{34}$$

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \leq 4e^{-\int_s^t \alpha(u)du} \|\mathbf{z}^*(s) - \mathbf{z}^{**}(s)\|_{1D}, \tag{35}$$

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1E} \leq \frac{2}{W} e^{-\int_s^t \alpha(u)du} \|\mathbf{z}^*(s) - \mathbf{z}^{**}(s)\|_{1D}, \tag{36}$$

for any initial conditions $s \geq 0$, $\mathbf{p}^*(s)$, $\mathbf{p}^{**}(s)$ and any $t \geq s$.

If in addition $D(\mathbf{p}^*(s) - \mathbf{p}^{**}(s)) \geq \mathbf{0}$, then

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1D} \geq e^{-\int_s^t \beta(u)du} \|\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\|_{1D}, \tag{37}$$

for any $0 \leq s \leq t$.

If the Markov chain is homogeneous, then all elements $b^*_{ij}(t)$ of the matrix $DB(t)D^{-1}$ do not dependent on t i.e. the quantities in (21) are constants. Thus instead of general bounds given by Theorem 1, one can specify then and obtain the following theorem.

Theorem 2. *Let there exist an increasing sequence* $\{d_j,\ j \geq 1\}$ *of positive numbers with* $d_1 = 1$*, such that (19) holds, and* $\alpha(t) = \alpha$ *defined by (21) is positive i.e.* $\alpha > 0$*. Then the Markov chain* $\{X(t),\ t \geq 0\}$ *is strongly ergodic and the following bounds hold:*

$$e^{-\chi t}\|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\|_{1D} \leq \|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1D} \leq \\ \leq e^{-\alpha t}\|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\|_{1D}, \tag{38}$$

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \leq 4e^{-\alpha t}\|\mathbf{z}^*(0) - \mathbf{z}^{**}(0)\|_{1D}, \tag{39}$$

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1E} \leq \frac{2}{W}e^{-\alpha t}\|\mathbf{z}^*(0) - \mathbf{z}^{**}(0)\|_{1D}, \tag{40}$$

for any initial conditions $s \geq 0$*,* $\mathbf{p}^*(0)$*,* $\mathbf{p}^{**}(0)$ *and any* $t \geq 0$*.*

If in addition $D(\mathbf{p}^*(0) - \mathbf{p}^{**}(0)) \geq \mathbf{0}$*, then*

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1D} \geq e^{-\beta t}\|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\|_{1D}, \tag{41}$$

for any $t \geq 0$*.*

For the decay parameter α^* *defined as*

$$\lim_{t\to\infty}(p_{ij}(t) - \pi_j) = O(e^{-\alpha^* t}),$$

where $\{\pi_j,\ j \geq 0\}$ *are the stationary probabilities of the chain, it holds that* $\alpha^* \geq \alpha$*.*

Notice that some additional results related to *Theorem* 2 can also be found in [4, 6]. If one assumes that the intensities $q_{ij}(t)$ are $1-$periodic in t i.e. $q_{ij}(t)$ are periodic functions and the length of the period is equal to one, then the Markov chain $\{X(t),\ t \geq 0\}$ has the limiting $1-$periodic limiting regime. Under the assumptions of Theorem 1 the Markov chain $\{X(t),\ t \geq 0\}$ is exponentially weakly ergodic. The detailed discussion of this results is given in [27].

Consider now a bit more detailed analysis of two special cases: homogeneous case and the case with periodic intensities. Firstly note that in the both cases there exist positive M and a such that

$$e^{-\int_s^t \alpha(u)\,du} \leq Me^{-a(t-s)} \tag{42}$$

for any $0 \leq s \leq t$. Hence the Markov chain $\{X(t),\ t \geq 0\}$ is *exponentially* weakly ergodic. Indeed, if the Markov chain $\{X(t),\ t \geq 0\}$ is homogeneous, then one may put $M = 1$, $a = \alpha$ given by (21). If all the intensity functions $q_{ij}(t)$ are $1-$periodic in t, then one may put

$$a=\int_0^1 \alpha(t)\,dt, \quad M=e^K, \quad K=\sup_{|t-s|\le 1}\int_s^t \alpha(u)\,du.$$

By doing so, for any solution of (5) the following bound holds:

$$\begin{aligned} \|\mathbf{z}(t)\|_{1D} \le \\ \|V(t)\|_{1D}\|\mathbf{z}(0)\|_{1D}+\int_0^t \|V(t,\tau)\|_{1D}\|\mathbf{f}(\tau)\|_{1D}\,d\tau \le \\ Me^{-at}\|\mathbf{z}(0)\|_{1D}+\frac{FM}{a}, \end{aligned} \tag{43}$$

where F is such that $\|\mathbf{f}(t)\|_{1D} \le F$ for almost all $t \in [0,1]$. Hence one has the upper bound for the limit

$$\limsup_{t\to\infty} \|\mathbf{z}(t)\|_{1D} \le \frac{FM}{a}, \tag{44}$$

for any initial condition and

$$\|\mathbf{p}(0)-\mathbf{e}_0\|_{1D}=\|\mathbf{p}(0)\|_{1D}=\|\mathbf{z}(0)\|_{1D} \le \limsup_{t\to\infty}\|\mathbf{z}(t)\|_{1D}, \tag{45}$$

where $\mathbf{e}_i$ denotes the unit vector of zeros with 1 in the i-th place. If the initial distribution is $\mathbf{p}^{**}(0)=\mathbf{e}_0$ then $\mathbf{z}^{**}(0)=\mathbf{0}$, $\mathbf{z}(t)\ge 0$ for any $\mathbf{p}^*(0)$ and any $t \ge 0$. Therefore

$$\begin{aligned} \|\mathbf{z}(t)\|_{1D} = d_1p_1+(d_1+d_2)p_2+ \\ +(d_1+d_2+d_3)p_3+\ldots = \\ = d_1p_1+\frac{d_1+d_2}{2}2p_2+\frac{d_1+d_2+d_3}{3}3p_3+\ldots \ge \\ \ge \inf_k \frac{d_1+\ldots+d_k}{k}\|\mathbf{z}(t)\|_{1E}, \end{aligned}$$

and one can use $W^*=\inf_k \frac{d_1+\ldots+d_k}{k}$ instead of $W=\inf_k \frac{d_k}{k}$, given by (5) in all the bounds on the rate of convergence. Finally, for the considered two special cases one has the following two corollaries.

Corollary 1. *Let $\{X(t),\ t\ge 0\}$ be a homogeneous Markov chain and let there exist an increasing sequence $\{d_j,\ j\ge 1\}$ of positive numbers with $d_1=1$ such that (19) holds and in addition $\alpha>0$. Then the Markov chain $\{X(t),\ t\ge 0\}$ is exponentially ergodic and the following bounds hold:*

$$\|\pi-\mathbf{p}(t,0)\| \le \frac{4F}{\alpha}e^{-\alpha t}, \tag{46}$$

$$|\varphi-E(t,0)| \le \frac{F}{\alpha W^*}e^{-\alpha t}, \tag{47}$$

where $\pi=(\pi_0,\pi_1,\ldots)^T$ denotes the vector of stationary probabilities of the chain and $\varphi=\sum_{j=0}^\infty j\pi_j$ and $\mathbf{p}(0,0)=\mathbf{e}_0$.

Corollary 2. *Assume that all the intensity functions of the Markov chain* $\{X(t),\ t \geq 0\}$ *are* $1-$*periodic in* t. *Let there exist an increasing sequence* $\{d_j,\ j \geq 1\}$ *of positive numbers with* $d_1 = 1$ *such that (19) holds and in addition* $\int_0^1 \alpha(t)\,dt = a > 0$. *Then the Markov chain* $\{X(t),\ t \geq 0\}$ *is exponentially weakly ergodic and the following bounds hold:*

$$\|\pi(t) - \mathbf{p}(t,0)\| \leq \frac{4FM}{a} e^{-at}, \tag{48}$$

$$|\varphi(t) - E(t,0)| \leq \frac{FM}{aW^*} e^{-at}, \tag{49}$$

where $\pi(t) = (\pi_0(t), \pi_1(t), \ldots)^T$ *denotes the vector of limiting probabilities of the chain and* $\varphi(t) = \sum_{j=0}^{\infty} j\pi_j(t)$ *and* $\mathbf{p}(0,0) = \mathbf{e}_0$.

If the state space of the Markov chain is finite there exist a number of special results (see [4, 6, 29]).

PERTURBATION BOUNDS.

Let $\{\bar{X}(t), t \geq 0\}$ be a perturbed Markov chain with transposed intensity matrix $\bar{A}(t)$ and the same restrictions as imposed on $A(t)$. It is assumed that $\|\hat{A}(t)\| = \|A(t) - \bar{A}(t)\| \leq \varepsilon$, for almost all $t \geq 0$, which means the perturbations are considered to be small in l_1 norm.

We can obtain the corresponding perturbation bounds. There are two different approaches.

The first approach in this direction are given in [8, 23] both for the discrete and continuous time Markov chains respectively. In the considered situation of Markov chains with continuous time, this approach is based on a comparison of the Cauchy operators of two linear equations in a Banach space considered in [2]. Consider equation (5) for the perturbed chain:

$$\frac{d}{dt}\bar{\mathbf{z}}(t) = \bar{B}(t)\bar{\mathbf{z}}(t) + \bar{\mathbf{f}}(t). \tag{50}$$

In this case, the weight space l_{1D} is considered as the base one, and the norms of perturbations are assumed to be small both in l_1 and l_{1D} norms. Namely, we suppose that $\|\hat{B}(t)\|_{1D} = \|B(t) - \bar{B}(t)\|_{1D} \leq \varepsilon$, and $\|\mathbf{f}(t) - \bar{\mathbf{f}}(t)\|_{1D} \leq \varepsilon$, for almost all $t \geq 0$.

The corresponding general results have been obtained in [32]. A typical statement of this kind is as follows:

Theorem 3. *Let the assumptions of Theorem 1 be fulfilled, and let, an addition,* $X(t)$ *be exponentially weakly ergodic in* l_{1D} *norm with the corresponding parameters* M_D, a_D *in (42). Then the following perturbation bound holds:*

$$\limsup_{t\to\infty} \|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\|_1 \leq \frac{4M_D\varepsilon\,(M_D\mathsf{F} + a_D)}{a_D\,(a_D - M_D\varepsilon)}, \tag{51}$$

where $\|\bar{\mathbf{f}}(t)\|_{1D} \leq \mathsf{F}$ *for almost all* $t \geq 0$.

The second approach also began with [23], namely, Mitrophanov [14] successfully applied probabilistic considerations and ergodicity in uniform operator topology which allowed to significantly reduce the constant factor in the stability estimate. The corresponding bounds for inhomogeneous situation has been obtained in [28].

A typical statement of this kind is as follows:

Theorem 4. *Let Markov cain $X(t)$ be exponentially weakly ergodic in l_1 norm with the corresponding parameters M^*, α^* in (42). Then the following bound holds:*

$$\limsup_{t\to\infty} \|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\|_1 \le \frac{\varepsilon\,(1+\log M^*)}{\alpha^*}. \qquad (52)$$

TRUNCATION BOUNDS.

Calculation of the limiting characteristics for (inhomogeneous) birth-death processes via truncations was firstly mentioned in [24] and was considered in details in [27]. First results for more general Markovian queueing models have been obtained recently in [31]. The respective bound of approximation error as a rule depends on time. Vladimir V. Kalashnikov in the early 1990-s suggested that in some cases one can obtain uniform (in time) error bounds of truncation. Such bounds for inhomogeneous birth-death processes have been obtained in [30], and for more general Markov chains in [33], this statement can be formulated in the following way.

Let $X_{N-1}(t)$ be a truncated process with the state space $\mathrm{E}_{N-1} = \{0,1,\dots,N-1\}$ and the corresponding transposed infinitesimal matrix

$$A_{N-1}(t) = \begin{pmatrix} b_{00}(t) & a_{01}(t) & \cdots & a_{0,N-1}(t) \\ a_{10}(t) & b_{11}(t) & \cdots & a_{1,N-1}(t) \\ a_{20}(t) & a_{21}(t) & \cdots & a_{2,N-1}(t) \\ \cdots & \cdots & \cdots & \cdots \\ a_{N-1,0}(t) & a_{N-1,1}(t) & \cdots & b_{N-1,N-1}(t) \end{pmatrix},$$

where $b_{ii}(t) = -\sum_{k=0,k\neq i}^{N-1} a_{ki}(t)$.

Then the forward Kolmogorov system for $X_{N-1}(t)$ is

$$\frac{d\mathbf{p}^*}{dt} = A_{N-1}(t)\mathbf{p}^*,$$

and instead of (5) we have

$$\frac{d\mathbf{z}^*}{dt} = B_{N-1}(t)\mathbf{z}^*(t) + \mathbf{f}_{N-1}(t), \qquad (53)$$

where $\mathbf{f}_{N-1}(t) = (a_{10}(t), a_{20}(t), \dots, a_{N-1,0}(t))^\top$, $\mathbf{z}^*(t) = (p_1, p_2, \cdots, p_{N-1})^\top$,

$$B_{N-1} = \begin{pmatrix} b_{11}(t)-a_{10}(t) & a_{12}(t)-a_{10}(t) & \cdots & a_{1,N-1}(t)-a_{10}(t) \\ a_{21}(t)-a_{20}(t) & b_{22}(t)-a_{20}(t) & \cdots & a_{2,N-1}(t)-a_{20}(t) \\ \cdots & \cdots & \cdots & \cdots \\ a_{N-1,1}(t)-a_{N-1,0}(t) & a_{N-1,2}(t)-a_{N-1,0}(t) & \cdots & b_{N-1,N-1}(t)-a_{N-1,0}(t) \end{pmatrix}.$$

Below we will identify the finite vector with entries $(a_1,\dots,a_{N-1})^\top$ and the infinite vector with the same first $N-1$ coordinates and the others equal to zero. Moreover we suppose that

$$a_{i+k,i}(t) = q_{i,i+k}(t) \le R\cdot q^{-k}, \qquad q>1, \quad R>0, \tag{54}$$

for any $k\ge 1, i\ge 0$ and almost all $t\ge 0$. For $\delta\in(1,\sqrt{q})$ we consider the sequences $d_k=\delta^{k-1}$ and $d_k^*=\delta^{2k-2}$, $k\ge 1$.

Denote

$$W=\inf_{i\ge 1}\frac{d_i}{i}, \qquad g_i=\sum_{n=1}^{i} d_n.$$

Let D and D^* be upper triangular matrices:

$$D=\begin{pmatrix} d_1 & d_1 & d_1 & \cdots \\ 0 & d_2 & d_2 & \cdots \\ 0 & 0 & d_3 & \cdots \\ & \ddots & \ddots & \ddots \end{pmatrix}, \quad D^*=\begin{pmatrix} d_1^* & d_1^* & d_1^* & \cdots \\ 0 & d_2^* & d_2^* & \cdots \\ 0 & 0 & d_3^* & \cdots \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

and l_{1D}, l_{1D^*} be the corresponding spaces of sequences:

$$\begin{aligned} l_{1D} &= \{\mathbf{z}=(p_1,p_2,\dots)^\top \,|\, \|\mathbf{z}\|_{1D}\equiv\|D\mathbf{z}\|_1<\infty\},\\ l_{1D^*} &= \{\mathbf{z}=(p_1,p_2,\dots)^\top \,|\, \|\mathbf{z}\|_{1D^*}\equiv\|D^*\mathbf{z}\|_1<\infty\}. \end{aligned}$$

We suppose that there exist positive constants M, a, M^*, a^* such that the following bounds

$$\|V(t,s)\|_{1D}\le Me^{-a(t-s)}, \tag{55}$$

and

$$\|V(t,s)\|_{1D^*}\le M^*e^{-a^*(t-s)}, \tag{56}$$

hold for Cauchy operator $V(t,s)$ of equation (5) for any s,t $(0\le s\le t)$. These estimates guarantee exponential convergence to zero as $t-s\to\infty$ of the difference $\|\mathbf{p}^*(t)-\mathbf{p}^{**}(t)\|\to 0$ in l_{1D} and l_{1D^*} norms respectively for the corresponding initial conditions.

Theorem 5. *Let the assumptions* (54), (55), (56) be fulfilled. *Then the following bounds of error of truncations hold:*

$$\|\mathbf{p}(t)-\mathbf{p}_{N-1}(t)\|\le C_1\left(\frac{\delta^2}{q}\right)^{N/3}+C_2\delta^{-N/3}+C_3\left(\frac{\delta}{q}\right)^{N} \tag{57}$$

and

$$|E(t,0)-E_{N-1}(t,0)|\le\frac{1}{W}\left\{C_1\left(\frac{\delta^2}{q}\right)^{N/3}+C_2\delta^{-N/3}+C_3\left(\frac{\delta}{q}\right)^{N}\right\}, \tag{58}$$

where index $N-1$ shows the corresponding characteristics of truncated process and $X(0) = X_{N-1}(0) = 0$. Moreover, constants $C_i = C_i(\delta, q)$ do not depend on N and on $\delta \in (1, \sqrt{q})$.

Finally we can briefly describe the possible procedure for finding $\pi(t)$ and $\varphi(t)$ in case of $1-$periodic in t intensities. Firstly we estimate the instant $t = t^*$ (using the ergodicity bounds), starting from which the solution of the forward Kolmogorov system (3) with the initial condition $X(0)$ is within the fixed $\varepsilon > 0$ from the limiting periodic probabilities. Then we estimate the size n^* of the state space $\{0, 1, \dots, n^*\}$, which guarantees the desired approximation error on the interval $[0, t^* + 1]$. Then we find the solution of the truncated system on the interval $[0, t^* + 1]$, eventually the values for $\pi(t)$ and $\varphi(t)$ on the interval $[t^*, t^* + 1]$.

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