Convergence Rate Estimates for Some Models of Queuing Theory, and Their Applications

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Abstract The forward Kolmogorov system for a general nonstationary Markovian queueing model with possible batch arrivals, possible catastrophes and state-dependent control at idle time is considered. We obtain upper bounds on the rate of convergence for corresponding models (nonstationary $M^X/M_n/1$ queue without catastrophes with the special resurrection intensities and general nonstationary $M^X/M_n/1$ queue with mass arrivals and catastrophes) and apply these estimates for some specific situations. Examples with given parameters are considered and corresponding plots are constructed.

1 Introduction

We consider forward Kolmogorov system for general nonstationary Markovian queueing model with possible batch arrivals, possible catastrophes and state-dependent control at idle time. The previous investigations in this area deal with different particular classes of this general model, see, for instance, [1–4, 6, 10]. Detailed discussion and references one can find in [4]. A general description of the model and basic results are given in [8]. Here we obtain upper bounds on the rate of convergence and apply them for some specific situations.

Let $X(t)$ be the queue-length process for this model. Denote by $\mathbf{p}(t)$ the column vector of state probabilities, $\mathbf{p}(t) = (p_0(t), p_1(t), \dots)^T$.

Then the probabilistic dynamics of the process $\{X(t),\ t \geq 0\}$ is given by the forward Kolmogorov system

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$$\frac{d\mathbf{p}(t)}{dt} = A(t)\mathbf{p}(t), \tag{1}$$

where

$$A(t) = \begin{pmatrix} q_{00}(t) & \beta_1(t) + \mu_1(t) & \beta_2(t) & \dots & \beta_j(t) & \dots & \\ h_1(t) & q_{11}(t) & \mu_2(t) & 0 & \dots & \dots & \dots \\ h_2(t) & b_1(t) & q_{22}(t) & \mu_3(t) & 0 & \dots & \dots \\ \vdots & \dots & \dots & \dots & \dots & \dots & \dots \\ \vdots & \dots & \dots & b_1(t) & q_{jj}(t) & \mu_{j+1}(t) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is the transposed intensity matrix, with the following non-zero entries of $A(t)$:

$b_k(t)$ are the intensity of arrival of group of k customers to the non-empty queue, which does not depend on the current size of the length of queue;
$\mu_k(t)$ are the intensity of service of a customer in the queue, if the current size of the length of queue equals k;
$\beta_k(t)$ are the disaster (catastrophe) intensity, if the current size of the length of queue equals k;
$h_k(t)$ are the intensity of transition from zero to k (resurrection in terms of [4], or mass arrivals in terms of [1]);
$q_{kk}(t)$ are such that the corresponding column sums of $A(t)$ are zero for any $t \geq 0$.

Note that all "intensity" functions $b_k(t)$, $\mu_k(t)$, $\beta_k(t)$ and $h_k(t)$ are nonnegative for any $t \geq 0$, locally integrable on $[0, \infty)$, and bounded on this interval, namely, that $|q_{kk}(t)| \leq L < \infty$ for almost all $t \geq 0$.

Then, applying the modified combined approach of [5] and [7] we can obtain bounds on the rate of convergence of the queue-length process to its limiting characteristics and compute them. We separately consider the important special cases, see description in [9] and general results in [8].

Throughout the paper by $\|\cdot\|$ we denote the l_1-norm, i. e. $\|\mathbf{p}(t)\| = \sum_k |p_k(t)|$, and $\|A(t)\| = \sup_j \sum_i |a_{ij}|$. Let Ω be a set all stochastic vectors, i.e. l_1 vectors with non-negative coordinates and unit norm. Hence the operator function $A(t)$ from l_1 into itself is bounded for almost all $t \geq 0$ and locally integrable on $[0; \infty)$, moreover $\|A(t)\| = 2\sup_k |q_{kk}(t)| \leq 2L$ for almost all $t \geq 0$. Therefore we can consider (1) as a differential equation in the space l_1 with bounded operator, hence the Cauchy problem for differential Eq. (1) has a unique solutions for an arbitrary initial condition, and $\mathbf{p}(s) \in \Omega$ implies $\mathbf{p}(t) \in \Omega$ for $t \geq s \geq 0$.

Denote by $E(t, k) = E(X(t)|X(0) = k)$ the conditional expected number of customers in the system at instant t, provided that initially (at instant $t = 0$) k customers were present in the system.

Recall that a Markov chain $\{X(t),\ t \geq 0\}$ is called *weakly ergodic*, if $\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \to 0$ as $t \to \infty$ for any initial conditions $\mathbf{p}^*(0)$ and $\mathbf{p}^{**}(0)$, where $\mathbf{p}^*(t)$ and $\mathbf{p}^{**}(t)$ are the corresponding solutions of (1). A Markov chain $\{X(t),\ t \geq 0\}$ has the limiting mean $\varphi(t)$, if $\lim_{t\to\infty} (\varphi(t) - E(t,k)) = 0$ for any k.

2 Nonstationary $M^X/M_n/1$ Queue Without Catastrophes with the Special Resurrection Intensities

In this section we study as in [4] the queueing model without catastrophes (i.e. all $\beta_j(t) = 0$) with the special resurrection rates $h_j(t) = b_j(t)$, for any j, t. In addition, we suppose in this section that $b_{k+1}(t) \leq b_k(t)$ for all k.

In accordance with these assumptions, we arrive at the model described in [7] as queue with state-independent batch arrivals and state-dependent service intensities.

Let $\{d_i\}$ be a sequence of positive numbers, and D be an upper triangular matrix,

$$D = \begin{pmatrix} d_1 & d_1 & d_1 & \cdots \\ 0 & d_2 & d_2 & \cdots \\ 0 & 0 & d_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Denote by $\mathbf{y}(t) = D\mathbf{z}(t)$, where $\mathbf{z}(t) = \left(p_1^*(t) - p_1^{**}(t), p_2^*(t) - p_2^{**}(t), \dots\right)^T$ is the difference of two solutions of the forward Kolmogorov system (1) with the corresponding initial conditions $\mathbf{p}^*(0)$ and $\mathbf{p}^{**}(0)$, in which all coordinates except p_0 are taken.

Put

$$\alpha_j(t) = \mu_j(t) - \frac{d_{j-1}}{d_j}\mu_{j-1}(t) + \sum_{i=1}^{\infty}\left(1 - \frac{d_{i+j}}{d_j}\right) b_i(t), \tag{2}$$

and

$$\alpha(t) = \inf \alpha_j(t). \tag{3}$$

Putting $p_0 = 1 - \sum_{i\geq 1} p_i$ and applying the logarithmic norm of operator function, see Theorem 1 in [8] and comparison of norms in [9], we get the following statement.

Proposition 1. *Let there exist an increasing sequence $\{d_j,\ j \geq 1\}$ of positive numbers with $d_1 = 1$, such that*

$$\int_0^\infty \alpha(t)\, dt = +\infty. \tag{4}$$

Then the Markov chain $X(t)$ is weakly ergodic and the following bound holds:

$$\|\mathbf{y}(t)\| \le e^{-\int_0^t \alpha(u)du} \|\mathbf{y}(0)\|, \tag{5}$$

and

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \le 4e^{-\int_0^t \alpha(u)du} \|\mathbf{y}(0)\|, \tag{6}$$

for any initial conditions $\mathbf{p}^*(0)$, $\mathbf{p}^{**}(0)$ and any $t \ge 0$.

Moreover, if $W = \inf_{i\ge 1} \frac{d_i}{i} > 0$, then $X(t)$ has the limiting mean and

$$|\varphi(t) - E(t,k)| \le \frac{2}{W} e^{-\int_0^t \alpha(u)du} \|\mathbf{y}(0)\|, \tag{7}$$

for any $t \ge 0$, and any k.

Here we apply all the bounds for nonstationary $M^X/M/S$ queue with batch arrivals and S servers, which is described and firstly studied in [7]. In this model we have the following intensities: $b_k(t) = \frac{1}{k}\lambda(t)$ if $1 \le k \le S$, and $b_k(t) = 0$ if $k > S$ are rates of arrival of a group of k customers; and $\mu_k(t) = \min(k, S)\mu(t)$ is the corresponding service rate.

Put $d_1 = 1$ and $d_{k+1} = \delta d_k$. Then

$$\alpha_k(t) = k\mu(t) - \frac{k-1}{\delta}\mu(t) + \sum_{i=1}^{S} \left(1 - \delta^i\right)\lambda(t), \tag{8}$$

if $k \le S$, and

$$\alpha_k(t) = S\mu(t)\left(1 - \frac{1}{\delta}\right) + \sum_{i=1}^{S} \left(1 - \delta^i\right)\lambda(t), \tag{9}$$

if $k > S$.

Denote $\Delta = \left(1 + (\delta+1)/2 + \cdots + \left(\delta^{S-1} + \cdots + \delta^2 + \delta + 1\right)/S\right)$, then we obtain

$$\alpha(t) = \min\left(\alpha_1(t), \alpha_{S+1}(t)\right) = \mu(t)\min\left(1, S - \frac{S}{\delta}\right) - \Delta(\delta - 1)\lambda(t). \tag{10}$$

Let now $\delta \in \left(1, \frac{S}{S-1}\right)$. Then $1 - \frac{1}{\delta} < \frac{1}{S}$ and hence

$$\alpha(t) = \left(1 - \delta^{-1}\right)\left(S\mu(t) - \Delta\delta\lambda(t)\right). \tag{11}$$

Then all assumptions of Proposition 1 for queue-length process of $M^X/M/S$ queue are fulfilled if

$$\int_0^{\infty} (S\mu(t) - \Delta\delta\lambda(t))\, dt = +\infty. \tag{12}$$

Let now arrival and service rates be 1-periodic in time.
Denote by $\lambda^* = \int_0^1 \lambda(t)\, dt$ and by $\mu^* = \int_0^1 \mu(t)\, dt$.
If $\delta = 1$ then $\Delta = S$ and $S\mu(t) - \Delta\delta\lambda(t) = S\,(\mu(t) - \lambda(t))$.
Therefore, if $\mu^* > \lambda^*$ then $S\mu^* - \Delta\delta\lambda^* > 0$, if $\delta - 1 > 0$ is small enough.
Finally, in 1-periodic situation the assumptions of Proposition 1 hold if $\mu^* > \lambda^*$.

Example 1. Consider the $M^X/M/S$ queue with $S = 10, \mu(t) = \mu = 3, \lambda(t) = 1 + M \sin 2\pi\omega t$ and different values of amplitude M, frequency ω.

One can put $\delta = 1.1$, then $e^{-\int_0^t \alpha(u)du} \le 2e^{-t}$ and $W > 0.23$. Here all assumptions of Proposition 1 hold and one can obtain the corresponding bounds on the rate of convergence to the limiting characteristics. One of the most important of them is the mean number of customers in the queue (the mathematical expectation).

The limiting mathematical expectation of the process and its dependence on the amplitude and frequency of the intensity of the arrival of requirements is shown (Figs. 1, 2, 3, 4, 5 and 6).

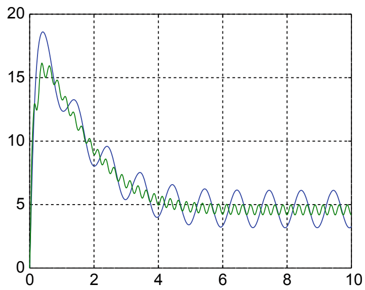

Fig. 1 Example 1. The mean $E(t, 0)$ for $t \in [0, 10]$ with $M = 1, \omega = 1$ (blue) and $M = 1, \omega = 4$ (green)

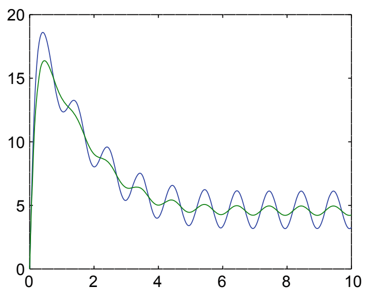

Fig. 2 Example 1. The mean $E(t, 0)$ for $t \in [0, 10]$ with $M = 1, \omega = 1$ (blue) and $M = 0.25, \omega = 1$ (green)

3 General Nonstationary $M^X/M_n/1$ Queue with Mass Arrivals and Catastrophes

Consider here more general situation. Let resurrection intensities $h_j(t)$ be arbitrary locally integrable functions such that $h_0(t) = \sum_{i\geq 1} h_i(t) \leq L$ in accordance with our general assumptions.

Rewrite the forward Kolmogorov system (1) as

$$\frac{d\mathbf{p}}{dt} = A^*(t)\,\mathbf{p} + \mathbf{g}(t), \quad t \geq 0, \tag{13}$$

where $\mathbf{g}(t) = (\beta_*(t), 0, 0, \dots)^T$, and $\beta_*(t) = \inf_i \beta_i(t)$. Then, applying the logarithmic norm of operator function, see Theorems 2 and 3 in [8], we get the following statements.

Proposition 2. *Let catastrophe rates be essential, i.e.*

$$\int_0^\infty \beta_*(t)\; dt = +\infty. \tag{14}$$

Then the queue-length process $X(t)$ is weakly ergodic in the uniform operator topology and the following bound holds

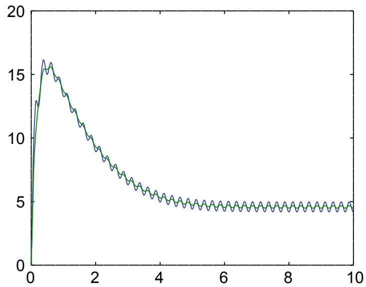

Fig. 3 Example 1. The mean $E(t, 0)$ for $t \in [0, 10]$ with $M = 1$, $\omega = 4$ (blue) and $M = 0.25$, $\omega = 4$ (green)

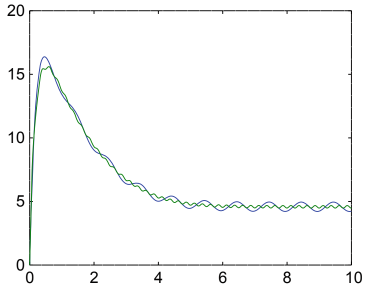

Fig. 4 Example 1. The mean $E(t, 0)$ for $t \in [0, 10]$ with $M = 0.25$, $\omega = 1$ (blue) and $M = 0.25$, $\omega = 4$ (green)

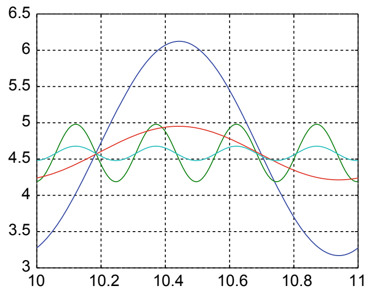

Fig. 5 Example 1. The mean $E(t,0)$ for $t \in [10, 11]$ for all four cases

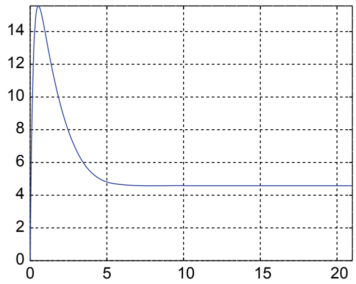

Fig. 6 Example 1. For comparison, the behaviour of the mean $E(t,0)$ for the process with constant service rate ($M = 0$) is shown here

$$\left\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\right\| \le e^{-\int\limits_0^t \beta_*(\tau)\,d\tau} \left\|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\right\| \le 2e^{-\int\limits_0^t \beta_*(\tau)\,d\tau}, \tag{15}$$

for any initial conditions $\mathbf{p}^*(0)$, $\mathbf{p}^{**}(0)$ *and any* $t \ge 0$.

Proposition 3. *Let* $\{d_i\}$, $1 = d_0 \le d_1 \le \ldots$ *be a non-decreasing sequence such that* $W = \inf_{i \ge 1} \frac{d_i}{i} > 0$, *and*

$$\int_0^\infty \beta_{**}(t)\,dt = +\infty, \tag{16}$$

where

$$\beta_{**}(t) = \inf_i \left(|a^*_{ii}(t)| - \sum_{j \ne i} \frac{d_j}{d_i} a^*_{ji}(t) \right), \tag{17}$$

and

$$a^*_{ij}(t) = \begin{cases} a_{0j}(t) - \beta_*(t), & \text{if } i = 0, \\ a_{ij}(t), & \text{otherwise}. \end{cases} \tag{18}$$

Then $X(t)$ has the limiting mean, say $\phi(t) = E(t,0)$, and the following bound holds:

$$|E(t,j) - E(t,0)| \le \frac{1+d_j}{W} e^{-\int_0^t \beta_{**}(\tau)\,d\tau}, \tag{19}$$

for any j and any $t \ge 0$.

Now we apply this approach for nonstationary $M^X/M/S$ queue with batch arrivals, S servers, possible resurrections and catastrophes. The corresponding results for these models for some situations were firstly obtained in [5, 6].

Consider the model with the following intensities: $b_k(t) = \frac{1}{k}\lambda(t)$ if $1 \le k \le S$, $b_k(t) = 0$ if $k > S$ are rates of arrival of a group of k customers; $\mu_k(t) = \min(k, S)\mu(t)$ is the corresponding service rate. In addition, we consider only general restrictions on the intensity of resurrection and catastrophe, namely we suppose that resurrection rates are decreasing exponentially: $h_k(t) \le cr^{-k}$ for some $r > 1$, $\beta_*(t) = \inf_i \beta_i(t)$.

Then the assumption of Proposition 2 is fulfilled if (14) hold.

Consider now the assumptions of Proposition 3. Put $d_0 = 1$ and $d_k = \delta^k$, where $\delta \in \left(1, \frac{S}{S-1}\right)$. Then we have $|a^*_{ii}(t)| - \sum_{j \ne i} \frac{d_j}{d_i} a^*_{ji}(t) \ge \beta_*(t) + \alpha(t)$, for $i \ge 1$, as in the previous Section. Let now $i = 0$. Then

$$|a^*_{00}(t)| - \sum_{j \ne 0} d_j a^*_{j0}(t) \ge \beta_*(t) - \sum_{k \ge 1} h_k(t)\left(\delta^k - 1\right) \ge \tag{20}$$

$$\beta_*(t) - c\sum_{k \ge 1} r^{-k}\left(\delta^k - 1\right) = \beta_*(t) - \frac{cr(\delta - 1)}{(r-\delta)(r-1)}, \tag{21}$$

hence

$$\beta_{**}(t) \ge \beta_*(t) - \frac{cr(\delta-1)}{(r-\delta)(r-1)}, \tag{22}$$

and (16) implies the validity of all the assumptions of Proposition 3.

Let now intensities be 1-periodic in time. Denote $\lambda^* = \int_0^1 \lambda(t)\,dt$, $\mu^* = \int_0^1 \mu(t)\,dt$, $\beta^*_* = \int_0^1 \beta_*(t)\,dt$.

In this situation assumption of Proposition 2 hold if $\beta^*_* > 0$, and if, in addition, $\mu^* > \lambda^*$ then Proposition 3 is also true.

Example 2. Consider the $M^X/M/S$ queue with batch arrivals, S servers, resurrections and catastrophes with the following parameters: $S = 10$, $\mu(t) = \mu = 2$, $\lambda(t) = 1 + M \sin 2\pi\omega t$, $\beta_k(t) = \frac{1}{2} + \frac{1}{k+1}(1 + \sin 2\pi t)$; $h_k(t) = 2^{1-k}(1 + \cos 2\pi t)$.

One can put here $\delta = 1.02$, then $\beta_*(t) \ge 0.5$,

$$\beta_{**}(t) \ge \beta_*(t) - \frac{cr(\delta-1)}{(r-\delta)(r-1)} \ge 0.5 - \frac{8 \cdot 1.02}{0.98} \ge 0.3,$$

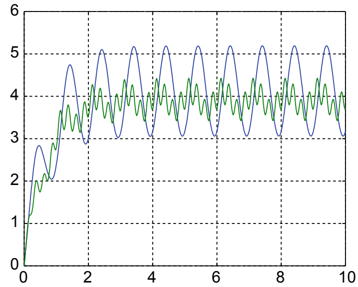

Fig. 7 Example 2. The mean $E(t, 0)$ for $t \in [0, 10]$ with $M = 1, \omega = 1$ (blue) and $M = 1, \omega = 4$ (green)

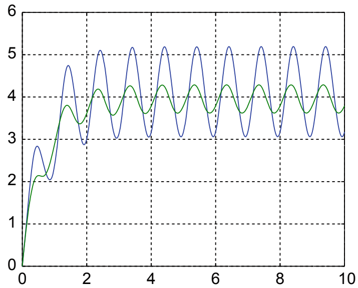

Fig. 8 Example 2. The mean $E(t, 0)$ for $t \in [0, 10]$ with $M = 1, \omega = 1$ (blue) and $M = 0.25, \omega = 1$ (green).

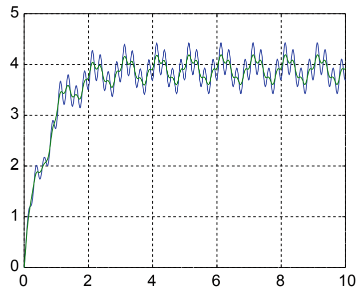

Fig. 9 Example 2. The mean $E(t, 0)$ for $t \in [0, 10]$ with $M = 1, \omega = 4$ (blue) and $M = 0.25, \omega = 4$ (green)

and $W \geq 0.05$. Hence all assumptions of Propositions 2, 3 hold and one can obtain the corresponding bounds on the rate of convergence to the limiting characteristics. One of the most important of them is the mean number of customers in the queue (the mathematical expectation).

The limiting mathematical expectation of the process and its dependence on the amplitude and frequency of the intensity of the arrival of requirements is shown (Figs. 7, 8, 9, 10, 11 and 12).

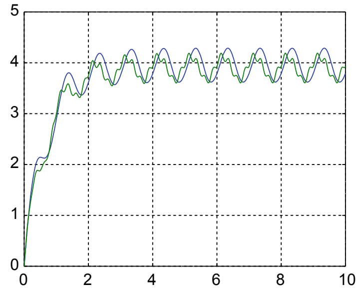

Fig. 10 Example 2. The mean $E(t, 0)$ for $t \in [0, 10]$ with $M = 0.25$, $\omega = 1$ (blue) and $M = 0.25$, $\omega = 4$ (green)

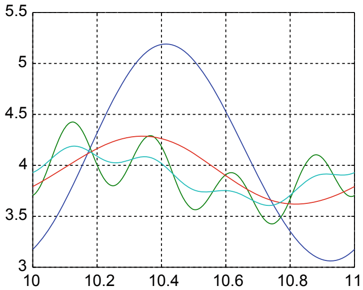

Fig. 11 Example 2. The mean $E(t, 0)$ for $t \in [10, 11]$ for all four cases

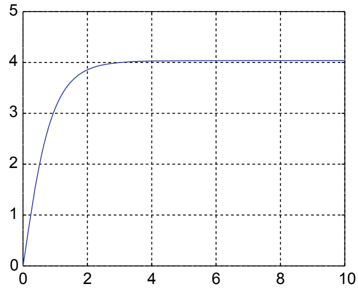

Fig. 12 Example 2. For comparison, the behaviour of the mean $E(t, 0)$ for the process with constant service rate ($M = 0$) is shown here

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Convergence rate estimates for some models of queuing theory, and their applications

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Abstract The forward Kolmogorov system for a general nonstationary Markovian queueing model with possible batch arrivals, possible catastrophes and state-dependent control at idle time is considered. We obtain upper bounds on the rate of convergence for corresponding models (nonstationary $M^X/M_n/1$ queue without catastrophes with the special resurrection intensities and general nonstationary $M^X/M_n/1$ queue with mass arrivals and catastrophes) and apply these estimates for some specific situations. Examples with given parameters are considered and corresponding plots are constructed.

1 Introduction

We consider forward Kolmogorov system for general nonstationary Markovian queueing model with possible batch arrivals, possible catastrophes and state-dependent control at idle time. The previous investigations in this area deal with different particular classes of this general model, see, for instance, [1, 2, 4, 5, 6, 8, 12]. Detailed discussion and references one can find in [5]. A general description of the model and preliminary results are given in [11]. Here we obtain upper bounds on the rate of convergence for such models and apply these estimates for some specific situations.

Let

$b_k(t)$ be the intensity of arrival of group of k customers to the non-empty queue, which does not depend on the current size of the length of queue;

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$\mu_k(t)$ be the intensity of service of a customer in the queue, if the current size of the length of queue equals k;

$\beta_k(t)$ be the disaster (catastrophe) intensity, if the current size of the length of queue equals k;

$h_k(t)$ be the intensity of transition from zero to k (resurrection in terms of [5], or mass arrivals in terms of [1]).

Consider the corresponding queue-length process $X(t)$. Then the intensity matrix $Q(t) = (q_{ij}(t))_{i,j=0}^{\infty}$ for $X(t)$ takes the following form:

$$Q(t) = \begin{pmatrix} q_{00}(t) & h_1(t) & h_2(t) & h_3(t) & h_4(t) & \dots & \dots \\ \beta_1(t)+\mu_1(t) & q_{11}(t) & b_1(t) & b_2(t) & \dots & \dots & \dots \\ \beta_2(t) & \mu_2(t) & q_{22}(t) & b_1(t) & b_2(t) & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \beta_j(t) & 0 & \dots & \mu_j(t) & q_{jj}(t) & b_1(t) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $q_{ii}(t)$ are such that all row sums of the matrix equal to zero for any $t \geq 0$.

Then the probabilistic dynamics of the process $\{X(t),\ t \geq 0\}$ is given by the forward Kolmogorov system

$$\frac{d\mathbf{p}(t)}{dt} = A(t)\mathbf{p}(t), \tag{1}$$

where

$$A(t) = Q^T(t) = \begin{pmatrix} q_{00}(t) & \beta_1(t)+\mu_1(t) & \beta_2(t) & \dots & \beta_j(t) & \dots & \\ h_1(t) & q_{11}(t) & \mu_2(t) & 0 & \dots & \dots & \dots \\ h_2(t) & b_1(t) & q_{22}(t) & \mu_3(t) & 0 & \dots & \dots \\ \vdots & \dots & \dots & \dots & \dots & \dots & \dots \\ \vdots & \dots & \dots & b_1(t) & q_{jj}(t) & \mu_{j+1}(t) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is the transposed intensity matrix, and $\mathbf{p}(t)$ is the column vector of state probabilities, $\mathbf{p}(t) = (p_0(t), p_1(t), \dots)^T$.

Then, applying the modified combined approach of [7] and [9] we can obtain bounds on the rate of convergence of the queue-length process to its limiting characteristics and compute them. We separately consider the important special cases.

2 Basic Notions

Denote by $p_{ij}(s,t) = P\{X(t) = j\,|X(s) = i\}$, $i, j \geq 0$, $0 \leq s \leq t$ the transition probabilities of $X(t)$ and by $p_i(t) = P\{X(t) = i\}$ – the probability that $X(t)$ is in state i at time t. Let $\mathbf{p}(t) = (p_0(t), p_1(t), \ldots)^T$ be probability distribution vector at instant t. Applying the standard approach (see for instance [8, 10]) we assume that all the intensity functions $q_{ij}(t)$ are locally integrable on $[0,\infty)$. Henceforth it is assumed that the intensity matrix $Q(t)$ is essentially bounded, i. e. $\sup_i |q_{ii}(t)| = L(t) \leq L < \infty$, for almost all $t \geq 0$.

Throughout the paper by $\|\cdot\|$ we denote the l_1-norm, i. e. $\|\mathbf{p}(t)\| = \sum_k |p_k(t)|$, and $\|A(t)\| = \sup_j \sum_i |a_{ij}|$. Let Ω be a set all stochastic vectors, i. e. l_1 vectors with non-negative coordinates and unit norm. Hence we have $\|A(t)\| = 2\sup_k |q_{kk}(t)| \leq 2L$ for almost all $t \geq 0$. Hence the operator function $A(t)$ from l_1 into itself is bounded for almost all $t \geq 0$ and locally integrable on $[0;\infty)$. Therefore we can consider (1) as a differential equation in the space l_1 with bounded operator.

It is well known (see [3]) that the Cauchy problem for differential equation (1) has a unique solutions for an arbitrary initial condition, and $\mathbf{p}(s) \in \Omega$ implies $\mathbf{p}(t) \in \Omega$ for $t \geq s \geq 0$.

Denote by $E(t,k) = E(X(t)|X(0) = k)$ the conditional expected number of customers in the system at instant t, provided that initially (at instant $t = 0$) k customers were present in the system.

Recall that a Markov chain $\{X(t),\ t \geq 0\}$ is called *weakly ergodic*, if $\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \to 0$ as $t \to \infty$ for any initial conditions $\mathbf{p}^*(0)$ and $\mathbf{p}^{**}(0)$, where $\mathbf{p}^*(t)$ and $\mathbf{p}^{**}(t)$ are the corresponding solutions of (1). A Markov chain $\{X(t),\ t \geq 0\}$ has the limiting mean $\varphi(t)$, if $\lim_{t\to\infty}(\varphi(t) - E(t,k)) = 0$ for any k.

3 Nonstationary $M^X/M_n/1$ Queue without Catastrophes with the Special Resurrection Intensities

In this section we study as in [5] the queueing model without catastrophes (i.e. all $\beta_j(t) = 0$) with the special resurrection rates $h_j(t) = b_j(t)$, for any j,t. In addition, we suppose in this section that $b_{k+1}(t) \leq b_k(t)$ for all k.

In accordance with these assumptions, we arrive at the model described in [10, 9] as queue with state-independent batch arrivals and state-dependent service intensities. Since $p_0(t) = 1 - \sum_{i=1}^{\infty} p_i(t)$ due to normalization condition, one can rewrite the system (1) as follows:

$$\frac{d\mathbf{z}(t)}{dt} = B(t)\mathbf{z}(t) + \mathbf{f}(t), \tag{2}$$

with $\mathbf{z}(t) = (p_1(t), p_2(t), \ldots)^T$, $\mathbf{f}(t) = (a_{10}(t), a_{20}(t), \ldots)^T$, and the corresponding $b_{ij}(t) = a_{ij}(t) - a_{i0}(t)$, $i, j \geq 1$.

Put now $\mathbf{y} = D(\mathbf{z}^* - \mathbf{z}^{**})$, where D is the upper triangular matrix:

$$D = \begin{pmatrix} d_1 & d_1 & d_1 & \cdots \\ 0 & d_2 & d_2 & \cdots \\ 0 & 0 & d_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then one has:

$$\frac{d\mathbf{y}(t)}{dt} = B_D(t)\mathbf{y}(t), \tag{3}$$

where

$$B_D(t) = \begin{pmatrix} a_{11}(t) & \frac{d_1}{d_2}\mu_1(t) & 0 & \cdots & 0 \\ \frac{d_2}{d_1}b_1(t) & a_{22}(t) & \frac{d_2}{d_3}\mu_2(t) & \cdots & 0 \\ \frac{d_3}{d_1}b_2(t) & \frac{d_3}{d_2}b_1(t) & a_{33}(t) & \frac{d_3}{d_4}\mu_3(t) & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{4}$$

is the corresponding reduced matrix $B_D(t) = DB(t)D^{-1} = b_{i,j,D}(t)$ with elements $b_{i,i+1,D}(t) = \frac{d_i}{d_{i+1}}\mu_i(t)$, $b_{i+k,i,D}(t) = \frac{d_{i+k}}{d_i}b_k(t)$, and all other off-diagonal elements equal zero for any $t \geq 0$. Hence matrix $B_D(t)$ is essentially nonnegative for any $t \geq 0$, therefore we can apply the notion of logarithmic norm of an operator function and related bounds.

Recall the respective notion and the corresponding bounds.

Namely, if $K(t)$, $t \geq 0$ is a one-parameter family of bounded linear operators on a Banach space $\mathscr{B}$, then

$$\gamma(K(t))_{\mathscr{B}} = \lim_{h \to +0} \frac{\|I + hK(t)\| - 1}{h} \tag{5}$$

is called the logarithmic norm of the operator $K(t)$. If $\mathscr{B} = l_1$ then the operator $K(t)$ is given by the matrix $K(t) = (k_{ij}(t))_{i,j=0}^{\infty}$, $t \geq 0$, and the logarithmic norm of $K(t)$ can be found explicitly:

$$\gamma(K(t)) = \sup_j \left(k_{jj}(t) + \sum_{i \neq j} \left| k_{ij}(t) \right| \right), \quad t \geq 0.$$

Then one has the following bound on the rate of convergence

$$\|\mathbf{x}(t)\| \leq e^{\int_0^t \gamma(K(\tau))d\tau} \|\mathbf{x}(0)\|,$$

if $\mathbf{x}(t)$ is an arbitrary solution of the differential equation

$$\frac{d\mathbf{x}(t)}{dt} = K(t)\mathbf{x}(t),$$

with the corresponding initial condition $\mathbf{x}(0)$.

Moreover, if the matrix $K(t)$ is essentially nonnegative then

$$\gamma(K(t)) = \sup_j \sum_i k_{ij}(t), \quad t \geq 0.$$

In this situation there is a number of classes of processes with possibility of sharp bounding the rate of convergence, see details in [7, 9, 10].

Put

$$\alpha_j(t) = \mu_j(t) - \frac{d_{j-1}}{d_j}\mu_{j-1}(t) + \sum_{i=1}^{\infty}\left(1 - \frac{d_{i+j}}{d_j}\right) b_i(t), \tag{6}$$

and

$$\alpha(t) = \inf \alpha_j(t). \tag{7}$$

Then we can compute

$$\gamma(B_D(t)) = \sup_j \sum_i b_{i,j,D}(t) = -\alpha(t).$$

Therefore the corresponding results of [10, 9] give us the following statement.

Theorem 1. *Let there exist an increasing sequence $\{d_j,\ j \geq 1\}$ of positive numbers with $d_1 = 1$, such that*

$$\int_0^{\infty} \alpha(t)\, dt = +\infty. \tag{8}$$

Then the Markov chain $X(t)$ is weakly ergodic and the following bound holds:

$$\|\mathbf{y}(t)\| \leq e^{-\int_s^t \alpha(u)du}\|\mathbf{y}(s)\|, \tag{9}$$

for any initial conditions $s \geq 0$, $\mathbf{p}^*(s)$, $\mathbf{p}^{**}(s)$ and any $t \geq s$. Moreover, if $W = \inf_{i\geq 1} \frac{d_i}{i} > 0$, then $X(t)$ has the limiting mean and

$$|\varphi(t) - E(t,k)| \leq \frac{2}{W} e^{-\int_s^t \alpha(u)du}\|\mathbf{y}(s)\|, \tag{10}$$

for any $s \geq 0$, $t \geq s$, and any k.

One can apply all the bounds for nonstationary $M^X/M/S$ queue with batch arrivals and S servers, which is described and studied in [9]. In this model we have the following intensities: $b_k(t) = \frac{1}{k}\lambda(t)$ if $1 \leq k \leq S$, and $b_k(t) = 0$ if $k > S$ are rates of arrival of a group of k customers; and $\mu_k(t) = \min(k,S)\mu(t)$ is the corresponding service rate. Let arrival and service rates be 1-periodic in time. Denote by $\lambda^* = \int_0^1 \lambda(t)\, dt$ and by $\mu^* = \int_0^1 \mu(t)\, dt$.

Then the assumptions of Theorem 1 hold if $\mu^* > \lambda^*$. For proof enough to put $d_1 = 1$ and $d_{k+1} = \delta d_k$, where $\delta \in \left(1, \frac{S}{S-1}\right)$. Then one has

$$\alpha_j(t) \geq \alpha(t) = \left(1-\delta^{-1}\right) \times \qquad (11)$$
$$\times \left\{S\mu(t) - \delta\lambda(t)\left(1+(\delta+1)/2+\ldots+\left(\delta^{S-1}+\delta+\delta+1\right)/S\right)\right\}.$$

If $\delta = 1$ then the expression in braces is equal to $S\mu(t) - S\lambda(t)$. Hence $\alpha^* = \int_0^1 \alpha(t)\,dt > 0$, if $\delta > 1$ is small enough, and the statement of Theorem 1 holds.

Example 1. Consider the $M^X/M/S$ queue with $S = 10$, $\mu(t) = \mu = 3$, $\lambda(t) = 1 + M\sin 2\pi\omega t$ and different values of amplitude M, frequency ω.

The limiting mathematical expectation of the process and its dependence on the amplitude and frequency of the intensity of the arrival of requirements is shown.

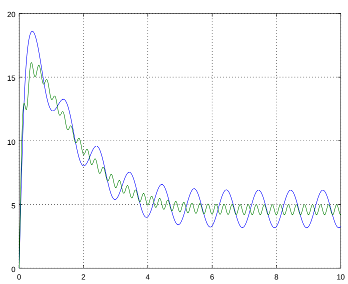

Fig. 1 Example 1. The mean $E(t,0)$ for $t \in [0,10]$ with $M = 1$, $\omega = 1$ (blue) and $M = 1$, $\omega = 4$ (green).

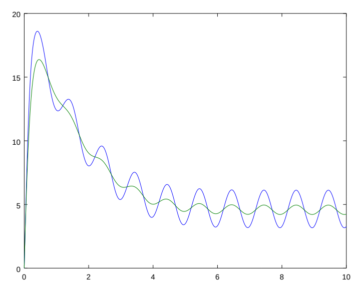

Fig. 2 Example 1. The mean $E(t,0)$ for $t \in [0,10]$ with $M = 1$, $\omega = 1$ (blue) and $M = 0.25$, $\omega = 1$ (green).

4 General Nonstationary $M^X/M_n/1$ Queue with Mass Arrivals and Catastrophes

Let now resurrection intensities $h_j(t)$ be arbitrary locally integrable functions such that $h_0(t) = \sum_{i\geq 1} h_i(t) \leq L$ in accordance with our general assumptions.

Let catastrophe intensities be *essential*, i. e. let

$$\int_0^\infty \beta_*(t)\,dt = +\infty, \qquad (12)$$

where $\beta_*(t) = \inf_i \beta_i(t)$.

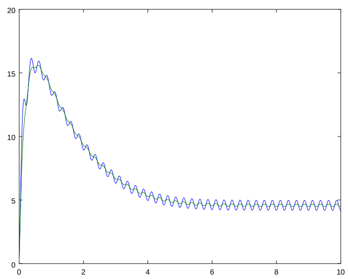

Fig. 3 Example 1. The mean $E(t,0)$ for $t \in [0,10]$ with $M = 1$, $\omega = 4$ (blue) and $M = 0.25$, $\omega = 4$ (green).

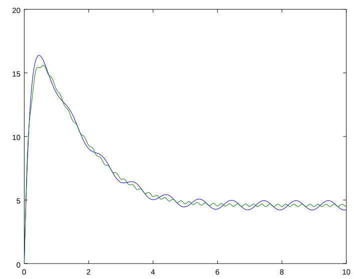

Fig. 4 Example 1. The mean $E(t,0)$ for $t \in [0,10]$ with $M = 0.25$, $\omega = 1$ (blue) and $M = 0.25$, $\omega = 4$ (green).

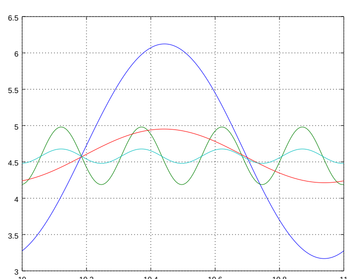

Fig. 5 Example 1. The mean $E(t,0)$ for $t \in [10,11]$ for all four cases.

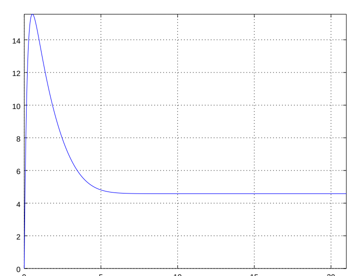

Fig. 6 Example 1. For comparison, here is shown behaviour of the mean $E(t,0)$ for the process with constant service rate, $M = 0$.

Rewrite the forward Kolmogorov system (1) as

$$\frac{d\mathbf{p}}{dt} = A^*(t)\,\mathbf{p} + \mathbf{g}(t), \quad t \geq 0. \tag{13}$$

Here $\mathbf{g}(t) = (\beta_*(t), 0, 0, \ldots)^T$, $A^*(t) = \left(a^*_{ij}(t)\right)_{i,j=0}^{\infty}$, and

$$a^*_{ij}(t) = \begin{cases} a_{0j}(t) - \beta_*(t), & \text{if } i = 0, \\ a_{ij}(t), & \text{otherwise}. \end{cases} \tag{14}$$

Put now $\mathbf{w} = \mathsf{D}(\mathbf{p}^* - \mathbf{p}^{**})$, where D is a diagonal matrix $\mathsf{D} = diag(d_0, d_1, d_2, \ldots)$ and $\{d_i\}$, $1 = d_0 \leq d_1 \leq \ldots$ is a non-decreasing sequence of positive numbers.

Then one has:

$$\frac{d\mathbf{w}(t)}{dt} = A^*_{\mathsf{D}}(t)\mathbf{w}(t), \tag{15}$$

where $A^*_{\mathsf{D}}(t) = \mathsf{D}A^*(t)\mathsf{D}^{-1} = a^*_{i,j,\mathsf{D}}(t)$ with the corresponding elements. Hence matrix $A^*_{\mathsf{D}}(t)$ is essentially nonnegative for any $t \geq 0$, therefore we can apply the notion of logarithmic norm of an operator function and related bounds.

Put firstly all $d_j = 1$.

Then we have

$$\gamma\left(A^*_{\mathsf{D}}(t)\right) = \sup_j \sum_i a^*_{i,j}(t) = -\beta_*(t).$$

Then the corresponding results of [6, 7, 8] give us the following statement.

Theorem 2. *Let catastrophe rates be essential, i. e. assumption (12) be fulfilled. Then the queue-length process $X(t)$ is weakly ergodic in the uniform operator topology and the following bound hold*

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \leq e^{-\int_0^t \beta_*(\tau)\,d\tau} \|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\| \leq 2e^{-\int_0^t \beta_*(\tau)\,d\tau}, \tag{16}$$

for any initial conditions $\mathbf{p}^*(0), \mathbf{p}^{**}(0)$ and any $t \geq 0$.

Consider now the bounds in ”weighted” norm.

Put

$$\alpha_i^{**}(t) = |a^*_{ii}(t)| - \sum_{j \neq i} \frac{d_j}{d_i} a^*_{ji}(t), \tag{17}$$

and

$$\beta_{**}(t) = \inf_i \alpha_i^{**}(t). \tag{18}$$

Then

$$\gamma\left(A^*_{\mathsf{D}}(t)\right) = \sup_i \sum_j \frac{d_j}{d_i} a^*_{j,i}(t) = -\beta_{**}(t),$$

and we obtain the following statement.

Theorem 3. *Let $\{d_i\}$, $1 = d_0 \le d_1 \le \dots$ be a non-decreasing sequence such that $W = \inf_{i\ge1} \frac{d_i}{i} > 0$, and*

$$\int_0^\infty \beta_{**}(t)\,dt = +\infty. \tag{19}$$

Then $X(t)$ has the limiting mean, say $\phi(t) = E(t,0)$, and the following bound holds:

$$|E(t,j) - E(t,0)| \le \frac{1+d_j}{W} e^{-\int\limits_0^t \beta_{**}(\tau)\,d\tau}, \tag{20}$$

for any j and any $t \ge 0$.

Now we apply this approach for nonstationary $M^X/M/S$ queue with batch arrivals, S servers, possible resurrections and catastrophes. The corresponding results for these models in the cases considered were firstly obtained in [6, 7, 8].

Consider the model with the following intensities: $b_k(t) = \frac{1}{k}\lambda(t)$ if $1 \le k \le S$, $b_k(t) = 0$ if $k > S$ are rates of arrival of a group of k customers; $\mu_k(t) = \min(k,S)\mu(t)$ is the corresponding service rate; resurrection rates are decreasing exponentially: $h_k(t) \le cr^{-k}$ for some $r > 1$, and $\beta_*(t) = \inf_i \beta_i(t)$.

Let all intensities be 1-periodic in time. Denote $\lambda^* = \int_0^1 \lambda(t)\,dt$, $\mu^* = \int_0^1 \mu(t)\,dt$, $\beta^*_* = \int_0^1 \beta_*(t)\,dt$.

Then the assumptions of Theorem 2 hold if $\beta^*_* > 0$. If, in addition, $\mu^* > \lambda^*$ then Theorem 3 is also true. To prove this, it suffices to put $d_0 = 1$ and $d_k = \delta^k$, where $\delta > 1$ and $\delta - 1$ is sufficiently small. Then $\alpha^{**}_j(t) \ge \beta_*(t) + \alpha(t)$, for $j \ge 1$, as in the previous Section. On the other hand,

$$\alpha^{**}_0(t) \ge \beta_*(t) - \sum_{k\ge1} h_k(t)\left(\delta^k - 1\right) \ge \beta_*(t) - c\sum_{k\ge1} r^{-k}\left(\delta^k - 1\right) =$$

$$= \beta_*(t) - \frac{cr(\delta-1)}{(r-\delta)(r-1)} = \beta_{**}(t).$$

Hence $\beta^*_{**} = \int_0^1 \beta_{**}(t)\,dt > 0$. Moreover, $W = \inf_{i\ge1} \frac{\delta^k}{k} > 0$, and all assumptions of Theorem 3 hold.

Example 2. Consider the $M^X/M/S$ queue with batch arrivals, S servers, resurrections and catastrophes with the following parameters: $S = 10$, $\mu(t) = \mu = 2$, $\lambda(t) = 1 + M\sin 2\pi\omega t$, $\beta_k(t) = \frac{1}{2} + \frac{1}{k+1}(1+\sin 2\pi t)$; $h_k(t) = 2^{1-k}(1+\cos 2\pi t)$.

The limiting mathematical expectation of the process and its dependence on the amplitude and frequency of the intensity of the arrival of requirements is shown.

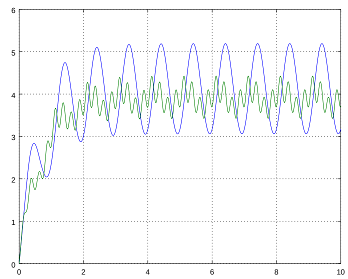

Fig. 7 Example 2. The mean $E(t,0)$ for $t \in [0,10]$ with $M = 1$, $\omega = 1$ (blue) and $M = 1$, $\omega = 4$ (green).

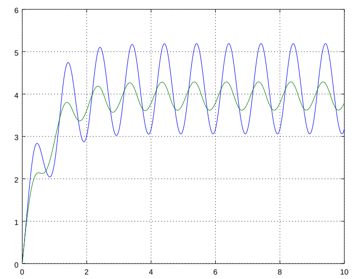

Fig. 8 Example 2. The mean $E(t,0)$ for $t \in [0,10]$ with $M = 1$, $\omega = 1$ (blue) and $M = 0.25$, $\omega = 1$ (green).

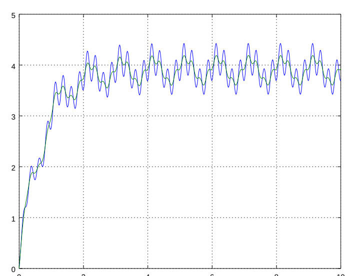

Fig. 9 Example 2. The mean $E(t,0)$ for $t \in [0,10]$ with $M = 1$, $\omega = 4$ (blue) and $M = 0.25$, $\omega = 4$ (green).

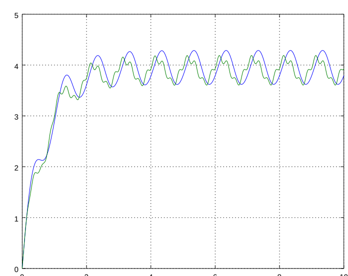

Fig. 10 Example 2. The mean $E(t,0)$ for $t \in [0,10]$ with $M = 0.25$, $\omega = 1$ (blue) and $M = 0.25$, $\omega = 4$ (green).

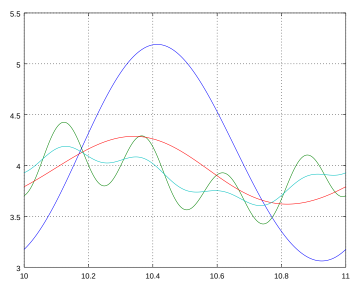

Fig. 11 Example 2. The mean $E(t,0)$ for $t \in [10,11]$ for all four cases.

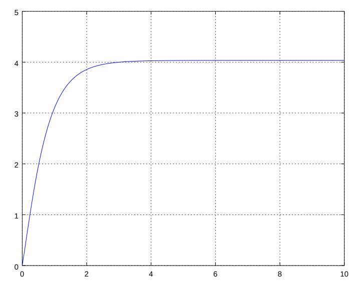

Fig. 12 Example 2. For comparison, here is shown behaviour of the mean $E(t,0)$ for the process with the corresponding constant rates.

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