Bounding the Rate of Convergence for One Class of Finite Capacity Time Varying Markov Queues

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Abstract. Consideration is given to the two finite capacity time varying Markov queues: the analogue of the well-known time varying $M/M/S/0$ queue with S servers each working at rate $\mu(t)$, no waiting line, but with the arrivals happening at rate $\lambda(t)$ only in batches of size 2; the analogue of another well-known time varying $M/M/1/(S-1)$ queue, but with the server, providing service at rate $\mu(t)$ if and only if there are at least 2 customers in the system, and with the arrivals happening only in batches of size 2. The functions $\lambda(t)$ and $\mu(t)$ are assumed to be non-random non-negative analytic functions of t. The new approach for the computation of the upper bound for the rate of convergence is proposed. It is based on the differential inequalities for the reduced forward Kolmogorov system of differential equations. Feasibility of the approach is demonstrated by the numerical example.

Keywords: Queueing systems · Rate of convergence · Non-stationary · Markovian queueing models · Limiting characteristics

1 Introduction

Non-stationary Markovian queueing models have been actively studied over the past few decades (see [1–6, 9, 15, 16, 19, 21] and the references therein) and the interest in this topic seems not to be declining. There exists one (to some extent)

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general framework for the analysis of such systems, which was developed in the series of papers by the authors. It consists of the following four steps[1]:

(a) find the upper bounds for the rate of convergence to the limiting regime[2];
(b) find the lower bounds for the rate of convergence to the limiting regime, which demonstrate that the dependence on the initial condition cannot vanish before some time instant t_*;
(c) obtain the stability (perturbation) bounds providing that if the structure of the rate (generator) matrix of the process is taken into account in an appropriate way, and the errors in the transition rates are small, then the basic characteristics of the process are calculated in an adequate way;
(d) approximate the process $X(t)$ by a similar, but truncated processes with a smaller number of states and construct the corresponding estimates for the approximation error.

By carrying out the steps (a)—(d) for the system with $1-$time-periodic rates and by solving the forward Kolmogorov system of differential equations (like (6)) with the simplest initial condition $X(0) = 0$ for the truncated process on the interval $[t^*, t^* + 1]$ one obtains all basic probability characteristics of the process $X(t)$ and the "perturbed" processes. It is worth noticing that the step (a) is the most important among the four. This is due to the fact that once the upper bounds are obtained all other steps (b)—(d) can be carried out in a straightforward manner, by using the results of [19–25].

It is worth noticing that exact estimates of the rate of convergence yield exact estimates of stability (perturbation) bounds (see [7,8,10–13,17,20] and references therein).

In the previous two papers [14,27] one has outlined the new approach for the computation of the upper bound for the rate of convergence, which is based on the application of differential inequalities to the reduced forward Kolmogorov system of differential equations. Here one presents the detailed description of this approach for the case of finite state inhomogeneous Markov chains (see Sect. 2). Its feasibility is demonstrated on one class of non-stationary Markovian queues (see Sect. 3). In the Sect. 4 the numerical example is given. Section 5 concludes the paper.

Throughout the paper vectors are regarded as column vectors, $^{\mathrm{T}}$ denotes the matrix transpose. The norm of a vector is denoted by $\|\cdot\|$ and means the

[1] For the more detailed description the reader is referred to [26].

[2] The limiting regime implies that beginning from a certain time instant, say, t^*, the probability characteristics of the process $X(t)$ for $t > t^*$ do not depend on the initial conditions (up to a given discrepancy). Note that a Markov chain $X(t)$ is called weakly ergodic, if $\|\boldsymbol{p}^*(t) - \boldsymbol{p}^{**}(t)\| \to 0$ as $t \to \infty$ for any initial conditions $\boldsymbol{p}^*(0)$ and $\boldsymbol{p}^{**}(0)$, where $\boldsymbol{p}^*(t)$ and $\boldsymbol{p}^{**}(t)$ are the corresponding solutions of (6). When considering weak ergodicity and inhomogeneous Markov chains, in general, any regime may be regarded as a limiting one. For example, in the case when the transition rates are $1-$periodic functions, the system (6) has $1-$periodic solution in the weak ergodic sense and it is reasonable to regard this solution as limiting.

sum of the absolute values of the vector's elements. When a vector, say $\boldsymbol{x}(t)$, is considered only for t from the fixed interval, say I, and not from the whole real positive line, the notation $\|\boldsymbol{x}(t)\|_I$ is used.

2 Description of the Approach

Consider a homogeneous system of linear differential equations in the vector-matrix form:

$$\frac{d}{dt}\boldsymbol{x}(t) = K(t)\boldsymbol{x}(t), \tag{1}$$

where $\boldsymbol{x}(t)$ is the real column vector and $K(t)$ is the $S \times S$ matrix with the elements $k_{ij}(t)$, being real functions, which are analytic for any $t \geq 0$. Let $\boldsymbol{x}(t)$ be the non-trivial solution of (1). Fix an arbitrary time instant $t = t_0$. Assume for now that $x_1(t_0) > 0$. Due to the continuity assumption for some value $\epsilon_1 > 0$ $x_1(t_0)$ remains positive in the interval $I_1 = (t_0 - \epsilon_1, t_0 + \epsilon_1)$. For other $S - 1$ elements of $\boldsymbol{x}(t_0)$ one can find other appropriate intervals $I_2, \ldots, I_S$ in which the sign of the corresponding element does not change. Denote by $I = (t_1, t_2)$ the intersection of all of these intervals i.e. $I = I_1 \cap \cdots \cap I_S$. In this common interval I the signs of the elements of $\boldsymbol{x}(t)$ do not change. Let us assume that $x_i(t) < 0$ for $i \in \{i_1, \ldots, i_k\} \subset \{1, \ldots, S\}$ and $x_i(t) \geq 0$ for $i \in \{1, \ldots, S\} \backslash \{i_1, \ldots, i_k\}$. Choose S positive numbers, say $\{d_1^I, \ldots, d_S^I\}$, and put $z_i(t) = -d_i^I x_i$ if $i \in \{i_1, \ldots, i_k\}$ and $z_i(t) = d_i^I x_i$ otherwise. Then $z_i(t) \geq 0$ for all $t \in I$ and $i \in \{1, \ldots, S\}$ and thus $\sum_{i=1}^S z_i(t)$ is the norm of the vector $\boldsymbol{z}(t)$ in the interval I. By differentiating $\|\boldsymbol{z}(t)\|_I$ with respect to t, one has:

$$\frac{d}{dt}\|\boldsymbol{z}(t)\|_I = \sum_{i=1}^S \frac{dz_i(t)}{dt} = \sum_{j=1}^S \underbrace{\sum_{i=1}^S \frac{d_i^I}{d_j^I}\vartheta_{ij}k_{ij}(t)}_{\alpha_j^I(t)} z_j(t) = \sum_{j=1}^S \alpha_j^I(t) z_j(t), \tag{2}$$

where $\vartheta_{ij} = 1$ if $x_i(t)$ and $x_j(t)$ are of the same sign and $\vartheta_{ij} = -1$ otherwise. Therefore from (2) one has the following upper bound

$$\frac{d}{dt}\|\boldsymbol{z}(t)\|_I \leq \alpha^I(t)\|\boldsymbol{z}(t)\|_I, \tag{3}$$

where $\alpha^I(t) = \max_{1 \leq j \leq S} \alpha_j^I(t)$ and thus

$$\|\boldsymbol{z}(\tau_2)\|_I \leq e^{\int_{\tau_1}^{\tau_2} \alpha^I(u)\, du}\|\boldsymbol{z}(\tau_1)\|_I,$$

for any $t_1 \leq \tau_1 \leq \tau_2 \leq t_2$. By comparing the norms $\|\boldsymbol{z}(t)\|_I$ and $\|\boldsymbol{x}(t)\|$ one obtains the following upper bound for the $\|\boldsymbol{x}(t)\|$:

$$\|\boldsymbol{x}(\tau_2)\| \leq C^I e^{\int_{\tau_1}^{\tau_2} \alpha^I(u)\, du}\|\boldsymbol{x}(\tau_1)\|, \tag{4}$$

for any $t_1 \leq \tau_1 \leq \tau_2 \leq t_2$, where $C^I = \frac{\max_{1 \leq i \leq S} d_i^I}{\min_{1 \leq i \leq S} d_i^I}$.

Note that the first step in the derivation of (4) was the assumption that some elements of $\boldsymbol{x}(t)$ are negative and the other are non-negative in I. But since the total number of elements in $\boldsymbol{x}(t)$ is S there are a total of 2^S such assumptions (i.e. 2^S possible combinations of elements' signs in $\boldsymbol{x}(t)$). Let us assume that for each of the 2^S combinations one can find proper I and $\{d_i^I, 1 \le i \le S\}$, and thereby compute $\alpha^I(t)$ and C^I. Thus one has 2^S upper bounds of type (4) and among them one can choose the worst one. Note that if for some t the two-sided derivative of $\|\boldsymbol{x}(t)\|$ does not exist, it can be replaced by the right-hand derivative. Thereby all possible combinations of elements' signs in $\boldsymbol{x}(t)$ are considered and the following theorem holds.

Theorem. *Let all $k_{ij}(t)$ be analytic functions of t for $t \ge 0$. Then for any $0 \le s \le t$ and any initial condition $\|\boldsymbol{x}(s)\|$ the following bound holds:*

$$\|\boldsymbol{x}(t)\| \le C e^{\int_s^t \alpha(u)\,du} \|\boldsymbol{x}(s)\|, \tag{5}$$

where $C = \max_{\text{all } I} C^I$, $\alpha(t) = \max_{\text{all } I} \alpha^I(t)$.

In the next section it is being demonstrated how this approach works in the case of several Markov queues with time varying arrival and service rates.

3 Model Description

Consider[3] a time varying $M/M/\cdot/S$ queue in which customers arrive only in the batches of size 2 with rate $\lambda(t)$. If a pair of customers arrives but there is no free room in the system for both customers, they both are lost. The service rate may depend on the total number of customers in the system and is equal to $\mu_i(t)$, when i customers are present in the system. Clearly, $\mu_0(t) = 0$. The functions $\lambda(t)$ and $\mu_i(t)$ are assumed to be non-random non-negative analytic functions of t.

In the notation $M/M/\cdot/S$ one has not specified the number of servers in the system. This is due to the fact that the number of servers explicitly depends on the values of $\mu_i(t)$. In what follows two extreme cases are considered:

(i) $\mu_i(t) = i\mu(t)$, $1 \le i \le S$, which means that the considered queue is the analogue of the well-known time varying $M/M/S/0$ queue with S servers each working at rate $\mu(t)$, no waiting line, but with the arrivals happening only in the batches of size 2;
(ii) $\mu_1(t) = 0$ and $\mu_i(t) = \mu(t)$, $2 \le i \le S$, which means that the considered queue is the variant of another well-known time varying $M/M/1/(S-1)$ queue, but this time with the server, providing service (at rate $\mu(t)$) if and only if there are at least 2 customers in the system, and with the arrivals happening only in the batches of size 2. Note that here only one customer at a time may be served.

[3] This is one of the four classes of systems considered in [24,25].

For the time being it is more convenient to assume that the service rate in the system is equal to $\mu_i(t)$ and do not specify which of the two cases, (i) or (ii), is being considered.

Let $X(t)$ be the Markov process, equal to the total number of customers in the system at time t i.e. $X(t)$ takes values in the finite set $\mathcal{X} = \{0, 1, \dots, S\}$. Denote by $p_{ij}(s,t) = P\{X(t) = j \,|X(s) = i\}$, $i, j \geq 0$, $0 \leq s \leq t$, the transition probabilities of $X(t)$ and by $p_i(t) = P\{X(t) = i\}$—the probability that $X(t)$ is in state i at time t. Let $\boldsymbol{p}(t) = (p_0(t), p_1(t), \dots, p_S(t))^{\mathrm{T}}$ be probability distribution vector at instant t. Throughout the paper it is assumed that in a small time interval h the possible transitions and their associated probabilities are

$$p_{ij}(t, t+h) = \begin{cases} q_{ij}(t)h + \alpha_{ij}\,(t,h)\,, & \text{if } j \neq i, \\ 1 - \sum\limits_{k \in \mathcal{X}, k \neq i} q_{ik}(t)h + \alpha_i\,(t,h)\,, & \text{if } j = i, \end{cases}$$

where $q_{ij}(t)$ are the transition rates and $\alpha_{ij}\,(t,h) = o(h)$ for all i, j. For the queueing system under consideration the transition rates can be easily specified: $q_{i,i+2}(t) = \lambda(t)$, $0 \leq i \leq S-2$, and $q_{i,i-1}(t) = \mu_i(t)$, $1 \leq i \leq S$.

The vector $\boldsymbol{p}(t)$ satisfies the forward Kolmogorov system of differential equations

$$\frac{d}{dt}\boldsymbol{p}(t) = A(t)\boldsymbol{p}(t), \tag{6}$$

where $A(t)$ is the transposed rate matrix i.e. $a_{ij}(t) = q_{ji}(t)$, $i, j \in \mathcal{X}$. Denote $\boldsymbol{f}(t) = (a_{10}(t), \dots, a_{S0}(t))^{\mathrm{T}}$ and $\boldsymbol{z}(t) = (p_1(t), \dots, p_S(t))^{\mathrm{T}}$ and introduce the new matrix[4] $B(t)$ of size $S \times S$, with the (i,j) entry $b_{ij}(t)$ equal to

$$b_{ij}(t) = a_{ij}(t) - a_{i0}(t), \quad 1 \leq i, j \leq S.$$

Using the normalization condition $p_0(t) = 1 - \sum_{i=1}^{S} p_i(t)$, the system (6) can rewritten as

$$\frac{d}{dt}\boldsymbol{z}(t) = B(t)\boldsymbol{z}(t) + \boldsymbol{f}(t).$$

All bounds of the rate of convergence to the limiting regime for $X(t)$ correspond to the same bounds of the solutions of the system

$$\frac{d}{dt}\boldsymbol{y}(t) = B(t)\boldsymbol{y}(t), \tag{7}$$

where $\boldsymbol{y}(t) = (y_1(t), \dots, y_S(t))^{\mathrm{T}}$ is the vector with the elements of arbitrary signs (not necessarily all non-negative as in $\boldsymbol{p}(t)$). As it was firstly noticed in [18], it is more convenient to study the rate of convergence using the transformed version $B(t)$ given by $B^*(t) = TB(t)T^{-1}$, where T is the $S \times S$ upper triangular matrix of the form

4 In other papers this matrix is sometimes called the reduced intensity matrix. It does not have any probabilistic interpretation.

$$T = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

After some algebraic manipulations it can be seen that for the queueing system under consideration the matrix $B^*(t)$ is equal[5] to

$$B^*(t) = \begin{pmatrix} -(\lambda(t)+\mu_1(t)) & \mu_1(t) & 0 & \ldots & 0 & 0 & 0 \\ 0 & -(\lambda(t)+\mu_2(t)) & \mu_2(t) & \ldots & 0 & 0 & 0 \\ \lambda(t) & 0 & -(\lambda(t)+\mu_3(t)) & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & -(\lambda(t)+\mu_{S-1}(t)) & \mu_{S-1}(t) \\ 0 & 0 & 0 & \ldots & \lambda(t) & -\lambda(t) & -\mu_S(t) \end{pmatrix}.$$

Introduce the new notation $\boldsymbol{u}(t) = T\boldsymbol{y}(t)$. Then the system (7) can be rewritten in the form

$$\frac{d}{dt}\boldsymbol{u}(t) = B^*(t)\boldsymbol{u}(t), \tag{8}$$

where $\boldsymbol{u}(t) = (u_1(t), \ldots, u_S(t))^{\mathrm{T}}$ is, as well as $\boldsymbol{y}(t)$, the vector with the elements of arbitrary signs (not necessarily all non-negative as in $\boldsymbol{p}(t)$). Notice that one has converted the system (6), describing the probabilistic dynamic of the total number of customers in the considered queue, to the system (8), which looks the same as (1). So (8) is the starting point for the application of the proposed method.

Consider the case (ii), which implies that the service rates $\mu_i(t)$ in the matrix $B^*(t)$ are equal to $\mu_1(t) = 0$ and $\mu_i(t) = \mu(t)$, $2 \le i \le S$. The sequence of steps by which one applies the method depends on whether the arrival rate is "larger" or "smaller" than the service rate. At the expense of some loss of generality[6] only the "larger" case is considered below. Let $\boldsymbol{u}(t)$ be the solution of (8). Remember that there are 2^S possible combinations of elements' signs in $\boldsymbol{u}(t)$. Assume that all elements of the $\boldsymbol{u}(t)$ are positive i.e. $u_i(t) > 0$, $1 \le i \le S$. Put $d_S = 1$, $d_{S-1} = \delta^{-1}$, $d_{S-2} = \delta$, and $d_{k-1} = \delta d_k$, for $1 \le k \le S-2$, where $\delta > 1$. Denoting $\boldsymbol{w}(t) = D\boldsymbol{u}(t)$, where $D = diag(d_1, \ldots, d_S)$, (8) can be rewritten in the form

$$\frac{d}{dt}\boldsymbol{w}(t) = B^{**}(t)\boldsymbol{w}(t),$$

[5] Note that whenever the matrix $B^*(t)$ after all these transformations turns out to be essentially non-negative for any $t \ge 0$ i.e. $b^*_{ij}(t) \ge 0$ for $i \ne j$, the rate of convergence can be studied using the logarithmic norm method (see [24,25]).

[6] Although the "smaller" case is not treated here, there is no principal difficulty, but longer sequence of steps in dealing with it. Note also that here the terms "larger" and "smaller" should be understood in the integral sense.

where $B^{**}(t) = DB^*(t)D^{-1}$. Let us write out the column sums of $B^{**}(t)$. For the sake of brevity introduce the notation $-\alpha_i(t) = \sum_{j=1}^{S} b_{ji}^{**}(t)$. Then

$$\begin{aligned}
\alpha_1(t) &= \lambda(t) - \delta^{-2}\lambda(t),\\
\alpha_2(t) &= (\lambda(t) + \mu(t)) - \delta^{-2}\lambda(t),\\
\alpha_k(t) &= (\lambda(t) + \mu(t)) - \delta^{-2}\lambda(t) - \delta\mu(t),\ 3 \le k \le S-3,\\
\alpha_{S-2}(t) &= (\lambda(t) + \mu(t)) - \delta^{-1}\lambda(t) - \delta\mu(t),\\
\alpha_{S-1}(t) &= (\lambda(t) + \mu(t)) + \delta\lambda(t) - \delta^2\mu(t),\\
\alpha_S(t) &= \mu(t) - \delta^{-1}\mu(t).
\end{aligned}$$

Hence for this interval one can bound the corresponding $\alpha^I(t)$ by

$$\alpha^I(t) = \min_{1\le i\le S} \alpha_i(t) = \left(1 - \delta^{-1}\right) \min\left(\mu(t), \lambda(t) - \delta\mu(t)\right). \tag{9}$$

The second argument in the $\min(\cdot,\cdot)$ function is positive since $\lambda(t)$ is assumed to be larger than $\mu(t)$. Assume now that $u_S(t) < 0$ and all other elements of $\boldsymbol{u}(t)$ are positive i.e. $u_i(t) > 0$, $1 \le i \le S-1$. Put $d_S = -1$, $d_{S-1} = \delta$ and $d_{k-1} = \delta d_k$, for $1 \le k \le S-1$, where $\delta > 1$. Then

$$\begin{aligned}
\alpha_1(t) &= \lambda(t) - \delta^{-2}\lambda(t),\\
\alpha_2(t) &= (\lambda(t) + \mu(t)) - \delta^{-2}\lambda(t),\\
\alpha_k(t) &= (\lambda(t) + \mu(t)) - \delta^{-2}\lambda(t) - \delta\mu(t),\ 3 \le k \le S-3,\\
\alpha_{S-2}(t) &= (\lambda(t) + \mu(t)) + \delta^{-2}\lambda(t) - \delta\mu(t),\\
\alpha_{S-1}(t) &= (\lambda(t) + \mu(t)) - \delta^{-1}\lambda(t) - \delta\mu(t),\\
\alpha_S(t) &= \mu(t) + \delta\mu(t).
\end{aligned}$$

Hence for this interval one can bound the corresponding $\alpha^I(t)$ by

$$\alpha^I(t) = \min_{1\le i\le S} \alpha_i(t) = \left(1 - \delta^{-1}\right) \left(\lambda(t) - \delta\mu(t)\right). \tag{10}$$

Moreover one can note that in all other $2^S - 2$ cases only negative elements in the columns of the matrix $B^*(t)$ can be added. Thus in all other intervals the values of $\alpha^I(t)$ is greater for the same $|d_k|$. Thus one obtains the following upper bound for the rate of convergence for the queueing system (ii):

$$\|\boldsymbol{u}(t)\| \le C^* e^{-\int_0^t \alpha^*(u)\,du} \|\boldsymbol{w}(0)\|, \tag{11}$$

for any $t \ge 0$, where $C^* = \delta^S$, $\alpha^*(t) = \left(1 - \delta^{-1}\right) \min\left(\mu(t), \lambda(t) - \delta\mu(t)\right)$. Moreover $X(t)$ is weakly ergodic and the following bound on the rate of convergence holds:

$$\|\boldsymbol{p}^*(t) - \boldsymbol{p}^{**}(t)\| \le 4C^* e^{-\int_0^t \alpha^*(u)\,du} \|\boldsymbol{w}(0)\|, \tag{12}$$

for any initial conditions.

Even though the case (i) can be treated in the same way as described above, it is more convenient to treat it differently. Notice that in the case (i) all off-diagonal elements of the matrix $B^*(t)$ are non-negative and the sums $\sum_{j=1}^{S} b_{ji}^*(t)$

for the matrix $B^*(t)$ are equal to $-\mu(t)$. Thus the logarithmic norm of the matrix $B^*(t)$ is $\gamma(B^*(t)) = -\mu(t)$ and one can apply the approach based on the notion of the logarithmic norm. The results from the papers [6,19,25] immediately give that $X(t)$ is weakly ergodic and the following bounds on the rate of convergence hold:

$$\|\boldsymbol{u}(t)\| \leq e^{-\int_0^t \mu(\tau)\, d\tau}\|\boldsymbol{u}(0)\|, \tag{13}$$

$$\|\boldsymbol{p}^*(t) - \boldsymbol{p}^{**}(t)\| \leq 4e^{-\int_0^t \mu(\tau)\, d\tau}\|\boldsymbol{u}(0)\|, \tag{14}$$

for any initial conditions.

4 Numerical Example

Using the proposed method one can calculate not only the rate of convergence but also the approximate values for the limiting performance characteristics of the process $X(t)$ for appropriate interval $[t_1, t_2]$ with the known approximation error (see steps (a)—(d) in the Sect. 1.)

Let in the queue considered in Sect. 3 the functions $\lambda(t)$ and $\mu(t)$ be $1-$periodic functions equal to $\lambda(t) = 4 + \sin(2\pi t)$ and $\mu(t) = 1 + \cos(2\pi t)$, respectively[7]. Let $S = 100$. Then by applying the convergence bounds[8] of the

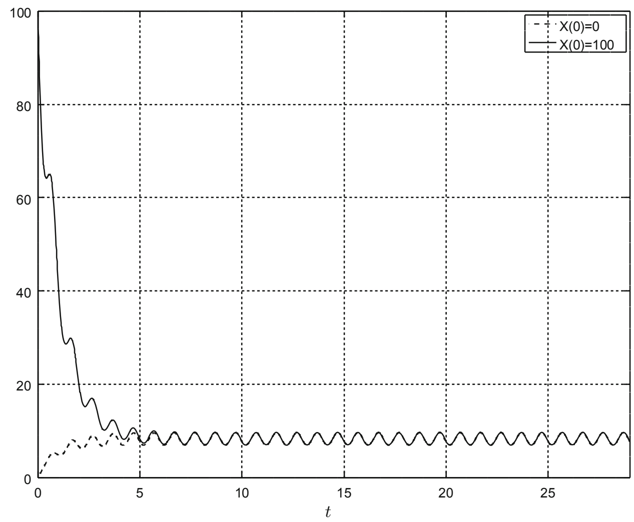

Fig. 1. Case (i). Rate of convergence of the mean number of customers in the system in the interval $[0, 30]$ for two different initial system occupancies ($X(0) = 0$ and $X(0) = 100$).

[7] Such choice of functions is justified as follows. Firstly, the results in Sect. 3 are presented for the case when $\lambda(t)$ is "larger" than $\mu(t)$. Secondly for $1-$periodic functions it is easier to decide which regime is reasonable to regard as a limiting one (see also the Footnote 2).

[8] For the case (i) the bound (14), for the case (ii) the bound (12).

previous section, one can compute, for example, the limiting value of the mean number of customers in the systems i.e. $\sum_{i=0}^{S} ip_i(t)$. For the case (i) in Fig. 1 one can see two graphs of the mean number of customers in the system at time t corresponding to two different initial conditions: when initially the system is empty (lower graph) and when initially the system is full (upper graph). The graphs are getting closer to each other as time t increases and eventually both coincide with the "limiting" graph, depicted in Fig. 2.

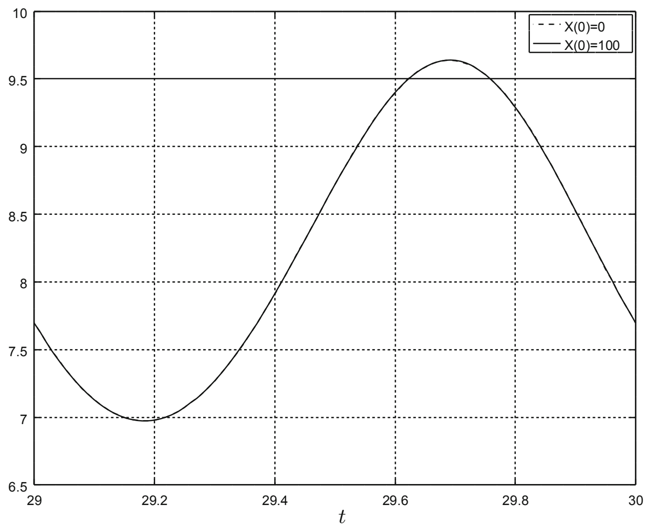

Fig. 2. Case (i). The limiting mean number of customers in the system in the interval $[29, 30]$ for two different initial system occupancies ($X(0) = 0$ and $X(0) = 100$).

Figures 3 and 4 provide the same information for the mean number of customers in the system for the case (ii). The time interval $[0, 30]$ (for both cases) was chosen by repeated attempts, shifting the right end of the interval until the convergence has become clearly visible. Note that by comparing Figs. 1 and 3 one can see that the convergence rate in the case (ii) is much slower than in the case (i).

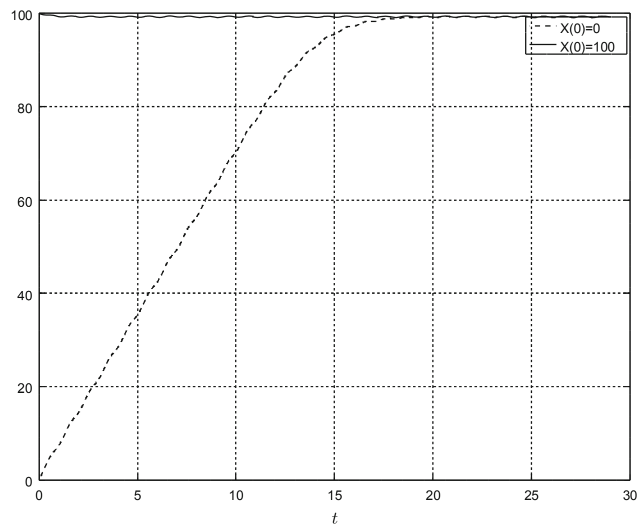

Fig. 3. Case (ii). Rate of convergence of the mean number of customers in the system in the interval $[0, 30]$ for two different initial system occupancies ($X(0) = 0$ and $X(0) = 100$).

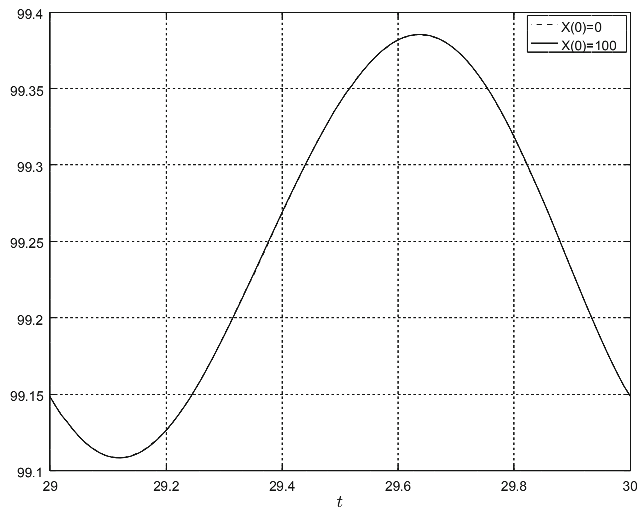

Fig. 4. Case (ii). The limiting mean number of customers in the system in the interval $[29, 30]$ for two different initial system occupancies ($X(0) = 0$ and $X(0) = 100$).

5 Conclusion

Coming back to (3) one can note that

$$\alpha^I(t) \le \max_{1\le j\le S}\left(k_{jj}(t) + \sum_{i=1,i\neq j}^{S} \frac{d_i^I}{d_j^I}|k_{ij}(t)|\right).$$

By putting $d_i^I = 1$ for all $1 \le i \le S$, one immediately arrives at the inequality $\alpha(t) \le \gamma(K(t))$, where $\gamma(K(t))$ is the logarithmic norm of the matrix $K(t)$. Thus the method proposed in Sect. 2 always gives results, which are no worse than results obtained using the approach based one the logarithmic norm. Since the logarithmic norm method gives exact bounds in the case of essential non-negativity of the matrix $K(t)$ (see [22]), the method of differential inequalities yields exact estimates in this case as well.

The proposed approach can be applied if and only if there is an opportunity to find proper constants $\{d_i^I, 1 \le i \le S\}$ for each interval I. Since (apparently) there does not exist any general algorithm for selecting $\{d_i^I, 1 \le i \le S\}$ for a general inhomogeneous birth and death process with a finite state space, the scope of the proposed approach is hard to define. For every new problem instance one has to examine the matrix $K(t)$ and act on the trial and error basis, when searching for $\{d_i^I, 1 \le i \le S\}$.

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Bounding the Rate of Convergence for One Class of Finite Capacity Time Varying Markov Queues*

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Abstract. Consideration is given to the two finite capacity time varying Markov queues: the analogue of the well-known time varying $M/M/S/0$ queue with S servers each working at rate $\mu(t)$, no waiting line, but with the arrivals happening at rate $\lambda(t)$ only in batches of size 2; the analogue of another well-known time varying $M/M/1/(S-1)$ queue, but with the server, providing service at rate $\mu(t)$ if and only if there are at least 2 customers in the system, and with the arrivals happening only in batches of size 2. The functions $\lambda(t)$ and $\mu(t)$ are assumed to be non-random non-negative analytic functions of t. The new approach for the computation of the upper bound for the rate of convergence is proposed. It is based on the differential inequalities for the reduced forward Kolmogorov system of differential equations. Feasibility of the approach is demonstrated by the numerical example.

Keywords: Queueing systems · Rate of convergence · Non-stationary · Markovian queueing models · Limiting characteristics.

1 Introduction

Non-stationary Markovian queueing models have been actively studied over the past few decades (see [1–6, 9, 15, 16, 19, 21] and the references therein) and the interest in this topic seems not to be declining. There exists one (to some extent) general framework for the analysis of such systems, which was developed in the series of papers by the authors. It consists of the following four steps[1]:

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[1] For the more detailed description the reader is referred to [26].

(a) find the upper bounds for the rate of convergence to the limiting regime[2];
(b) find the lower bounds for the rate of convergence to the limiting regime, which demonstrate that the dependence on the initial condition cannot vanish before some time instant t_*;
(c) obtain the stability (perturbation) bounds providing that if the structure of the rate (generator) matrix of the process is taken into account in an appropriate way, and the errors in the transition rates are small, then the basic characteristics of the process are calculated in an adequate way;
(d) approximate the process $X(t)$ by a similar, but truncated processes with a smaller number of states and construct the corresponding estimates for the approximation error.

By carrying out the steps (a)—(d) for the system with $1-$time-periodic rates and by solving the forward Kolmogorov system of differential equations (like (6)) with the simplest initial condition $X(0) = 0$ for the truncated process on the interval $[t^*, t^* + 1]$ one obtains all basic probability characteristics of the process $X(t)$ and the "perturbed" processes. It is worth noticing that the step (a) is the most important among the four. This is due to the fact that once the upper bounds are obtained all other steps (b)—(d) can be carried out in a straightforward manner, by using the results of [19]-[25].

It is worth noticing that exact estimates of the rate of convergence yield exact estimates of stability (perturbation) bounds (see [7, 8, 10–13, 17, 20] and references therein).

In the previous two papers [14, 27] one has outlined the new approach for the computation of the upper bound for the rate of convergence, which is based on the application of differential inequalities to the reduced forward Kolmogorov system of differential equations. Here one presents the detailed description of this approach for the case of finite state inhomogeneous Markov chains (see Section 2). Its feasibility is demonstrated on one class of non-stationary Markovian queues (see Section 3). In the Section 4 the numerical example is given. Section 5 concludes the paper.

Throughout the paper vectors are regarded as column vectors, $^{\mathrm{T}}$ denotes the matrix transpose. The norm of a vector is denoted by $\|\cdot\|$ and means the sum of the absolute values of the vector's elements. When a vector, say $\boldsymbol{x}(t)$, is considered only for t from the fixed interval, say I, and not from the whole real positive line, the notation $\|\boldsymbol{x}(t)\|_I$ is used.

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[2] The limiting regime implies that beginning from a certain time instant, say, t^*, the probability characteristics of the process $X(t)$ for $t > t^*$ do not depend on the initial conditions (up to a given discrepancy). Note that a Markov chain $X(t)$ is called weakly ergodic, if $\|\boldsymbol{p}^*(t) - \boldsymbol{p}^{**}(t)\| \to 0$ as $t \to \infty$ for any initial conditions $\boldsymbol{p}^*(0)$ and $\boldsymbol{p}^{**}(0)$, where $\boldsymbol{p}^*(t)$ and $\boldsymbol{p}^{**}(t)$ are the corresponding solutions of (6). When considering weak ergodicity and inhomogeneous Markov chains, in general, any regime may be regarded as a limiting one. For example, in the case when the transition rates are $1-$periodic functions, the system (6) has $1-$periodic solution in the weak ergodic sense and it is reasonable to regard this solution as limiting.

2 Description of the Approach

Consider a homogeneous system of linear differential equations in the vector-matrix form:

$$\frac{d}{dt}\boldsymbol{x}(t) = K(t)\boldsymbol{x}(t), \tag{1}$$

where $\boldsymbol{x}(t)$ is the real column vector and $K(t)$ is the $S \times S$ matrix with the elements $k_{ij}(t)$, being real functions, which are analytic for any $t \geq 0$. Let $\boldsymbol{x}(t)$ be the non-trivial solution of (1). Fix an arbitrary time instant $t = t_0$. Assume for now that $x_1(t_0) > 0$. Due to the continuity assumption for some value $\epsilon_1 > 0$ $x_1(t_0)$ remains positive in the interval $I_1 = (t_0 - \epsilon_1, t_0 + \epsilon_1)$. For other $S - 1$ elements of $\boldsymbol{x}(t_0)$ one can find other appropriate intervals $I_2, \dots, I_S$ in which the sign of the corresponding element does not change. Denote by $I = (t_1, t_2)$ the intersection of all of these intervals i.e. $I = I_1 \cap \cdots \cap I_S$. In this common interval I the signs of the elements of $\boldsymbol{x}(t)$ do not change. Let us assume that $x_i(t) < 0$ for $i \in \{i_1, \dots, i_k\} \subset \{1, \dots, S\}$ and $x_i(t) \geq 0$ for $i \in \{1, \dots, S\} \backslash \{i_1, \dots, i_k\}$. Choose S positive numbers, say $\{d_1^I, \dots, d_S^I\}$, and put $z_i(t) = -d_i^I x_i$ if $i \in \{i_1, \dots, i_k\}$ and $z_i(t) = d_i^I x_i$ otherwise. Then $z_i(t) \geq 0$ for all $t \in I$ and $i \in \{1, \dots, S\}$ and thus $\sum_{i=1}^S z_i(t)$ is the norm of the vector $\boldsymbol{z}(t)$ in the interval I. By differentiating $\|\boldsymbol{z}(t)\|_I$ with respect to t, one has:

$$\frac{d}{dt}\|\boldsymbol{z}(t)\|_I = \sum_{i=1}^S \frac{dz_i(t)}{dt} = \sum_{j=1}^S \underbrace{\sum_{i=1}^S \frac{d_i^I}{d_j^I}\vartheta_{ij}k_{ij}(t)}_{\alpha_j^I(t)}\, z_j(t) = \sum_{j=1}^S \alpha_j^I(t) z_j(t), \tag{2}$$

where $\vartheta_{ij} = 1$ if $x_i(t)$ and $x_j(t)$ are of the same sign and $\vartheta_{ij} = -1$ otherwise. Therefore from (2) one has the following upper bound

$$\frac{d}{dt}\|\boldsymbol{z}(t)\|_I \leq \alpha^I(t)\|\boldsymbol{z}(t)\|_I, \tag{3}$$

where $\alpha^I(t) = \max_{1\leq j\leq S} \alpha_j^I(t)$ and thus

$$\|\boldsymbol{z}(\tau_2)\|_I \leq e^{\int_{\tau_1}^{\tau_2} \alpha^I(u)\, du}\|\boldsymbol{z}(\tau_1)\|_I,$$

for any $t_1 \leq \tau_1 \leq \tau_2 \leq t_2$. By comparing the norms $\|\boldsymbol{z}(t)\|_I$ and $\|\boldsymbol{x}(t)\|$ one obtains the following upper bound for the $\|\boldsymbol{x}(t)\|$:

$$\|\boldsymbol{x}(\tau_2)\| \leq C^I e^{\int_{\tau_1}^{\tau_2} \alpha^I(u)\, du}\|\boldsymbol{x}(\tau_1)\|, \tag{4}$$

for any $t_1 \leq \tau_1 \leq \tau_2 \leq t_2$, where $C^I = \frac{\max_{1\leq i\leq S} d_i^I}{\min_{1\leq i\leq S} d_i^I}$.

Note that the first step in the derivation of (4) was the assumption that some elements of $\boldsymbol{x}(t)$ are negative and the other are non-negative in I. But since the total number of elements in $\boldsymbol{x}(t)$ is S there are a total of 2^S such assumptions (i.e. 2^S possible combinations of elements' signs in $\boldsymbol{x}(t)$). Let us assume that for each of the 2^S combinations one can find proper I and $\{d_i^I, 1 \le i \le S\}$, and thereby compute $\alpha^I(t)$ and C^I. Thus one has 2^S upper bounds of type (4) and among them one can choose the worst one. Note that if for some t the two-sided derivative of $\|\boldsymbol{x}(t)\|$ does not exist, it can be replaced by the right-hand derivative. Thereby all possible combinations of elements' signs in $\boldsymbol{x}(t)$ are considered and the following theorem holds.

Theorem. *Let all $k_{ij}(t)$ be analytic functions of t for $t \ge 0$. Then for any $0 \le s \le t$ and any initial condition $\|\boldsymbol{x}(s)\|$ the following bound holds:*

$$\|\boldsymbol{x}(t)\| \le C e^{\int_s^t \alpha(u)\,du} \|\boldsymbol{x}(s)\|, \tag{5}$$

where $C = \max_{\text{all } I} C^I$, $\alpha(t) = \max_{\text{all } I} \alpha^I(t)$.

In the next section it is being demonstrated how this approach works in the case of several Markov queues with time varying arrival and service rates.

3 Model Description

Consider[3] a time varying $M/M/\cdot/S$ queue in which customers arrive only in the batches of size 2 with rate $\lambda(t)$. If a pair of customers arrives but there is no free room in the system for both customers, they both are lost. The service rate may depend on the total number of customers in the system and is equal to $\mu_i(t)$, when i customers are present in the system. Clearly, $\mu_0(t) = 0$. The functions $\lambda(t)$ and $\mu_i(t)$ are assumed to be non-random non-negative analytic functions of t.

In the notation $M/M/\cdot/S$ one has not specified the number of servers in the system. This is due to the fact that the number of servers explicitly depends on the values of $\mu_i(t)$. In what follows two extreme cases are considered:

(i) $\mu_i(t) = i\mu(t)$, $1 \le i \le S$, which means that the considered queue is the analogue of the well-known time varying $M/M/S/0$ queue with S servers each working at rate $\mu(t)$, no waiting line, but with the arrivals happening only in the batches of size 2;

(ii) $\mu_1(t) = 0$ and $\mu_i(t) = \mu(t)$, $2 \le i \le S$, which means that the considered queue is the variant of another well-known time varying $M/M/1/(S-1)$ queue, but this time with the server, providing service (at rate $\mu(t)$) if and only if there are at least 2 customers in the system, and with the arrivals happening only in the batches of size 2. Note that here only one customer at a time may be served.

For the time being it is more convenient to assume that the service rate in the system is equal to $\mu_i(t)$ and do not specify which of the two cases, (i) or (ii), is being considered.

Let $X(t)$ be the Markov process, equal to the total number of customers in the system at time t i.e. $X(t)$ takes values in the finite set $\mathcal{X} = \{0, 1, \ldots, S\}$. Denote by $p_{ij}(s,t) = P\{X(t) = j \,|X(s) = i\}$, $i, j \ge 0$, $0 \le s \le t$, the transition

[3] This is one of the four classes of systems considered in [24, 25].

probabilities of $X(t)$ and by $p_i(t) = P\{X(t) = i\}$ — the probability that $X(t)$ is in state i at time t. Let $\boldsymbol{p}(t) = (p_0(t), p_1(t), \dots, p_S(t))^{\mathrm{T}}$ be probability distribution vector at instant t. Throughout the paper it is assumed that in a small time interval h the possible transitions and their associated probabilities are

$$p_{ij}(t, t+h) = \begin{cases} q_{ij}(t)h + \alpha_{ij}(t,h), & \text{if } j \neq i, \\ 1 - \sum\limits_{k \in \mathcal{X}, k \neq i} q_{ik}(t)h + \alpha_i(t,h), & \text{if } j = i, \end{cases}$$

where $q_{ij}(t)$ are the transition rates and $\alpha_{ij}(t,h) = o(h)$ for all i, j. For the queueing system under consideration the transition rates can be easily specified: $q_{i,i+2}(t) = \lambda(t)$, $0 \le i \le S-2$, and $q_{i,i-1}(t) = \mu_i(t)$, $1 \le i \le S$.

The vector $\boldsymbol{p}(t)$ satisfies the forward Kolmogorov system of differential equations

$$\frac{d}{dt}\boldsymbol{p}(t) = A(t)\boldsymbol{p}(t), \tag{6}$$

where $A(t)$ is the transposed rate matrix i.e. $a_{ij}(t) = q_{ji}(t)$, $i, j \in \mathcal{X}$. Denote $\boldsymbol{f}(t) = (a_{10}(t), \dots, a_{S0}(t))^{\mathrm{T}}$ and $\boldsymbol{z}(t) = (p_1(t), \dots, p_S(t))^{\mathrm{T}}$ and introduce the new matrix[4] $B(t)$ of size $S \times S$, with the (i,j) entry $b_{ij}(t)$ equal to

$$b_{ij}(t) = a_{ij}(t) - a_{i0}(t), \quad 1 \le i, j \le S.$$

Using the normalization condition $p_0(t) = 1 - \sum_{i=1}^{S} p_i(t)$, the system (6) can rewritten as

$$\frac{d}{dt}\boldsymbol{z}(t) = B(t)\boldsymbol{z}(t) + \boldsymbol{f}(t).$$

All bounds of the rate of convergence to the limiting regime for $X(t)$ correspond to the same bounds of the solutions of the system

$$\frac{d}{dt}\boldsymbol{y}(t) = B(t)\boldsymbol{y}(t), \tag{7}$$

where $\boldsymbol{y}(t) = (y_1(t), \dots, y_S(t))^{\mathrm{T}}$ is the vector with the elements of arbitrary signs (not necessarily all non-negative as in $\boldsymbol{p}(t)$). As it was firstly noticed in [18], it is more convenient to study the rate of convergence using the transformed version $B(t)$ given by $B^*(t) = TB(t)T^{-1}$, where T is the $S \times S$ upper triangular matrix of the form

$$T = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

[4] In other papers this matrix is sometimes called the reduced intensity matrix. It does not have any probabilistic interpretation.

After some algebraic manipulations it can be seen that for the queueing system under consideration the matrix $B^*(t)$ is equal[5] to

$$B^*(t) = \begin{pmatrix} -(\lambda(t)+\mu_1(t)) & \mu_1(t) & 0 & \dots & 0 & 0 & 0 \\ 0 & -(\lambda(t)+\mu_2(t)) & \mu_2(t) & \dots & 0 & 0 & 0 \\ \lambda(t) & 0 & -(\lambda(t)+\mu_3(t)) & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -(\lambda(t)+\mu_{S-1}(t)) & \mu_{S-1}(t) \\ 0 & 0 & 0 & \dots & \lambda(t) & -\lambda(t) & -\mu_S(t) \end{pmatrix}.$$

Introduce the new notation $\boldsymbol{u}(t) = T\boldsymbol{y}(t)$. Then the system (7) can be rewritten in the form

$$\frac{d}{dt}\boldsymbol{u}(t) = B^*(t)\boldsymbol{u}(t), \tag{8}$$

where $\boldsymbol{u}(t) = (u_1(t), \dots, u_S(t))^{\mathrm{T}}$ is, as well as $\boldsymbol{y}(t)$, the vector with the elements of arbitrary signs (not necessarily all non-negative as in $\boldsymbol{p}(t)$). Notice that one has converted the system (6), describing the probabilistic dynamic of the total number of customers in the considered queue, to the system (8), which looks the same as (1). So (8) is the starting point for the application of the proposed method.

Consider the case (ii), which implies that the service rates $\mu_i(t)$ in the matrix $B^*(t)$ are equal to $\mu_1(t) = 0$ and $\mu_i(t) = \mu(t)$, $2 \le i \le S$. The sequence of steps by which one applies the method depends on whether the arrival rate is "larger" or "smaller" than the service rate. At the expense of some loss of generality[6] only the "larger" case is considered below. Let $\boldsymbol{u}(t)$ be the solution of (8). Remember that there are 2^S possible combinations of elements' signs in $\boldsymbol{u}(t)$. Assume that all elements of the $\boldsymbol{u}(t)$ are positive i.e. $u_i(t) > 0$, $1 \le i \le S$. Put $d_S = 1$, $d_{S-1} = \delta^{-1}$, $d_{S-2} = \delta$, and $d_{k-1} = \delta d_k$, for $1 \le k \le S-2$, where $\delta > 1$. Denoting $\boldsymbol{w}(t) = D\boldsymbol{u}(t)$, where $D = diag(d_1, \dots, d_S)$, (8) can be rewritten in the form

$$\frac{d}{dt}\boldsymbol{w}(t) = B^{**}(t)\boldsymbol{w}(t),$$

where $B^{**}(t) = DB^*(t)D^{-1}$. Let us write out the column sums of $B^{**}(t)$. For the sake of brevity introduce the notation $-\alpha_i(t) = \sum_{j=1}^{S} b^{**}_{ji}(t)$. Then

$$\begin{aligned} \alpha_1(t) &= \lambda(t) - \delta^{-2}\lambda(t), \\ \alpha_2(t) &= (\lambda(t) + \mu(t)) - \delta^{-2}\lambda(t), \\ \alpha_k(t) &= (\lambda(t) + \mu(t)) - \delta^{-2}\lambda(t) - \delta\mu(t),\ 3 \le k \le S-3, \\ \alpha_{S-2}(t) &= (\lambda(t) + \mu(t)) - \delta^{-1}\lambda(t) - \delta\mu(t), \\ \alpha_{S-1}(t) &= (\lambda(t) + \mu(t)) + \delta\lambda(t) - \delta^2\mu(t), \\ \alpha_S(t) &= \mu(t) - \delta^{-1}\mu(t). \end{aligned}$$

[5] Note that whenever the matrix $B^*(t)$ after all these transformations turns out to be essentially non-negative for any $t \ge 0$ i.e. $b^*_{ij}(t) \ge 0$ for $i \ne j$, the rate of convergence can be studied using the logarithmic norm method (see [24, 25]).

[6] Although the "smaller" case is not treated here, there is no principal difficulty, but longer sequence of steps in dealing with it. Note also that here the terms "larger" and "smaller" should be understood in the integral sense.

Hence for this interval one can bound the corresponding $\alpha^I(t)$ by

$$\alpha^I(t) = \min_{1\le i\le S} \alpha_i(t) = \left(1-\delta^{-1}\right)\min\left(\mu(t), \lambda(t)-\delta\mu(t)\right). \tag{9}$$

The second argument in the $\min(\cdot,\cdot)$ function is positive since $\lambda(t)$ is assumed to be larger than $\mu(t)$. Assume now that $u_S(t) < 0$ and all other elements of $\boldsymbol{u}(t)$ are positive i.e. $u_i(t) > 0$, $1 \le i \le S-1$. Put $d_S = -1$, $d_{S-1} = \delta$ and $d_{k-1} = \delta d_k$, for $1 \le k \le S-1$, where $\delta > 1$. Then

$$\begin{aligned}
\alpha_1(t) &= \lambda(t) - \delta^{-2}\lambda(t),\\
\alpha_2(t) &= (\lambda(t)+\mu(t)) - \delta^{-2}\lambda(t),\\
\alpha_k(t) &= (\lambda(t)+\mu(t)) - \delta^{-2}\lambda(t) - \delta\mu(t),\ 3 \le k \le S-3,\\
\alpha_{S-2}(t) &= (\lambda(t)+\mu(t)) + \delta^{-2}\lambda(t) - \delta\mu(t),\\
\alpha_{S-1}(t) &= (\lambda(t)+\mu(t)) - \delta^{-1}\lambda(t) - \delta\mu(t),\\
\alpha_S(t) &= \mu(t) + \delta\mu(t).
\end{aligned}$$

Hence for this interval one can bound the corresponding $\alpha^I(t)$ by

$$\alpha^I(t) = \min_{1\le i\le S} \alpha_i(t) = \left(1-\delta^{-1}\right)\left(\lambda(t)-\delta\mu(t)\right). \tag{10}$$

Moreover one can note that in all other 2^S-2 cases only negative elements in the columns of the matrix $B^*(t)$ can be added. Thus in all other intervals the values of $\alpha^I(t)$ is greater for the same $|d_k|$. Thus one obtains the following upper bound for the rate of convergence for the queueing system (ii):

$$\|\boldsymbol{u}(t)\| \le C^* e^{-\int_0^t \alpha^*(u)\,du}\|\boldsymbol{w}(0)\|, \tag{11}$$

for any $t \ge 0$, where $C^* = \delta^S$, $\alpha^*(t) = \left(1-\delta^{-1}\right)\min\left(\mu(t), \lambda(t)-\delta\mu(t)\right)$. Moreover $X(t)$ is weakly ergodic and the following bound on the rate of convergence holds:

$$\|\boldsymbol{p}^*(t) - \boldsymbol{p}^{**}(t)\| \le 4C^* e^{-\int_0^t \alpha^*(u)\,du}\|\boldsymbol{w}(0)\|, \tag{12}$$

for any initial conditions.

Even though the case (i) can be treated in the same way as described above, it is more convenient to treat it differently. Notice that in the case (i) all off-diagonal elements of the matrix $B^*(t)$ are non-negative and the sums $\sum_{j=1}^S b^*_{ji}(t)$ for the matrix $B^*(t)$ are equal to $-\mu(t)$. Thus the logarithmic norm of the matrix $B^*(t)$ is $\gamma(B^*(t)) = -\mu(t)$ and one can apply the approach based on the notion of the logarithmic norm. The results from the papers [6, 19, 25] immediately give that $X(t)$ is weakly ergodic and the following bounds on the rate of convergence hold:

$$\|\boldsymbol{u}(t)\| \le e^{-\int_0^t \mu(\tau)\,d\tau}\|\boldsymbol{u}(0)\|, \tag{13}$$

$$\|\boldsymbol{p}^*(t) - \boldsymbol{p}^{**}(t)\| \le 4e^{-\int_0^t \mu(\tau)\,d\tau}\|\boldsymbol{u}(0)\|, \tag{14}$$

for any initial conditions.

4 Numerical Example

Using the proposed method one can calculate not only the rate of convergence but also the approximate values for the limiting performance characteristics of the process $X(t)$ for appropriate interval $[t_1, t_2]$ with the known approximation error (see steps (a)—(d) in the section 1.)

Let in the queue considered in Section 3 the functions $\lambda(t)$ and $\mu(t)$ be 1−periodic functions equal to $\lambda(t) = 4 + \sin(2\pi t)$ and $\mu(t) = 1 + \cos(2\pi t)$, respectively[7]. Let $S = 100$. Then by applying the convergence bounds[8] of the previous section, one can compute, for example, the limiting value of the mean number of customers in the systems i.e. $\sum_{i=0}^{S} ip_i(t)$. For the case (i) in Fig. 1 one can see two graphs of the mean number of customers in the system at time t corresponding to two different initial conditions: when initially the system is empty (lower graph) and when initially the system is full (upper graph). The graphs are getting closer to each other as time t increases and eventually both coincide with the "limiting" graph, depicted in Fig. 2.

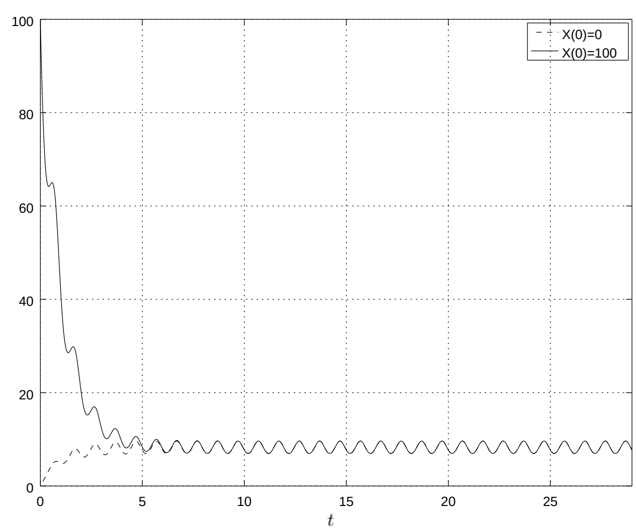

Fig. 1. Case (i). Rate of convergence of the mean number of customers in the system in the interval $[0, 30]$ for two different initial system occupancies ($X(0) = 0$ and $X(0) = 100$).

[7] Such choice of functions is justified as follows. Firstly, the results in Section 3 are presented for the case when $\lambda(t)$ is "larger" than $\mu(t)$. Secondly for 1−periodic functions it is easier to decide which regime is reasonable to regard as a limiting one (see also the footnote 2).

[8] For the case (i) the bound (14), for the case (ii) the bound (12).

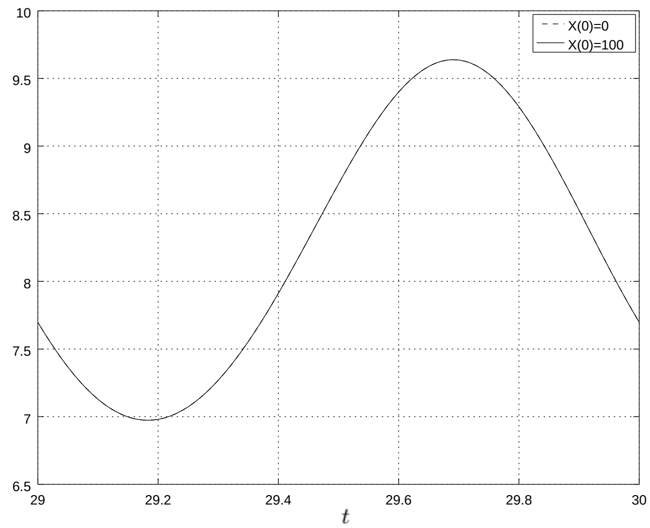

Fig. 2. Case (i). The limiting mean number of customers in the system in the interval $[29, 30]$ for two different initial system occupancies ($X(0) = 0$ and $X(0) = 100$).

Figures 3–4 provide the same information for the mean number of customers in the system for the case (ii). The time interval $[0, 30]$ (for both cases) was chosen by repeated attempts, shifting the right end of the interval until the convergence has become clearly visible. Note that by comparing Fig. 1 and Fig. 3 one can see that the convergence rate in the case (ii) is much slower than in the case (i).

5 Conclusion

Coming back to (3) one can note that

$$\alpha^I(t) \leq \max_{1\leq j\leq S}\left(k_{jj}(t) + \sum_{i=1, i\neq j}^{S} \frac{d_i^I}{d_j^I}|k_{ij}(t)|\right).$$

By putting $d_i^I = 1$ for all $1 \leq i \leq S$, one immediately arrives at the inequality $\alpha(t) \leq \gamma(K(t))$, where $\gamma(K(t))$ is the logarithmic norm of the matrix $K(t)$. Thus the method proposed in Section 2 always gives results, which are no worse than results obtained using the approach based one the logarithmic norm. Since the logarithmic norm method gives exact bounds in the case of essential non-negativity of the matrix $K(t)$ (see [22]), the method of differential inequalities yields exact estimates in this case as well.

The proposed approach can be applied if and only if there is an opportunity to find proper constants $\{d_i^I, 1 \leq i \leq S\}$ for each interval I. Since (apparently)

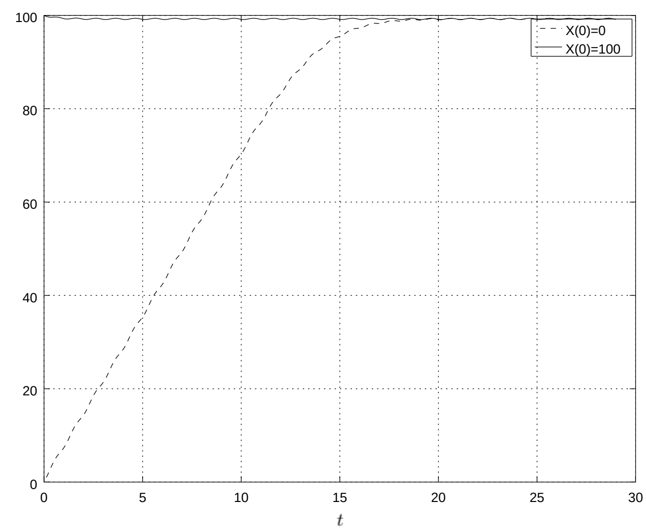

Fig. 3. Case (ii). Rate of convergence of the mean number of customers in the system in the interval $[0, 30]$ for two different initial system occupancies ($X(0) = 0$ and $X(0) = 100$).

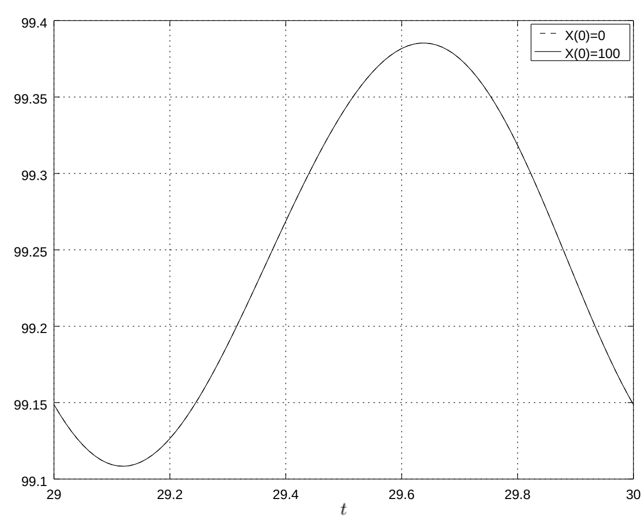

Fig. 4. Case (ii). The limiting mean number of customers in the system in the interval $[29, 30]$ for two different initial system occupancies ($X(0) = 0$ and $X(0) = 100$).

there does not exist any general algorithm for selecting $\{d_i^I, 1 \leq i \leq S\}$ for a general inhomogeneous birth and death process with a finite state space, the scope of the proposed approach is hard to define. For every new problem instance one has to examine the matrix $K(t)$ and act on the trial and error basis, when searching for $\{d_i^I, 1 \leq i \leq S\}$.

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