Application of Method of Differential Inequalities to Bounding the Rate of Convergence for a Class of Markov Chains

Anastasia Kryukova, Victoria Oshushkova, Alexander Zeifman, and Yacov Satin

Abstract We consider the linear system of differential equations $\frac{d\mathbf{p}}{dt} = A(t)\mathbf{p}$, which is the forward Kolmogorov system, for a class of Markov chains with 'batch' births and single deaths. We apply the method of differential inequalities for obtaining bounds on the rate of convergence for the system. A specific queueing model is considered and the corresponding limiting characteristics are computing.

Keywords Forward Kolmogorov system · Markov chains

1 Introduction and General Bounds

Let $\{X(t),\ t \geq 0\}$ be a continuous-time Markov chain with finite state space $\mathcal{X} = \{0, 1, \dots, N\}$. Denote by $p_{ij}(s,t) = P\{X(t) = j\,|X(s) = i\}$, $i, j \geq 0,\ 0 \leq s \leq t$ the transition probabilities of $X(t)$ and by $p_i(t) = P\{X(t) = i\}$ – the probability that the Markov chain $X(t)$ is in state i at time t. Let $\mathbf{p}(t) = (p_0(t), p_1(t), \dots)^T$ be the vector of state probabilities at the moment t.

Then the probabilistic dynamics of the process $\{X(t),\ t \geq 0\}$ is described by the forward Kolmogorov system

$$\frac{d\mathbf{p}}{dt} = A(t)\mathbf{p}, \tag{1}$$

where $A(t) = Q^T(t)$ is the transposed intensity matrix. All column sums of this matrix are zeros for any $t \geq 0$, and $A(t)$ is essentially nonnegative (i.e. all its off-diagonal elements are nonnegative for any $t \geq 0$).

We suppose that all 'intensity functions' $a_{ij}(t)$ are analytic in t for $t \geq 0$.

Consider a queueing model for a queue with batch arrivals and single services, see the first motivation in [2] and more recent studies in [1, 5, 6].

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Then we have $a_{ij}(t) = 0$ for $i < j-1$, all arrival rates do not depend on the size of a queue, i.e. $a_{i+k,i}(t) = a_k(t)$ for $k \geq 1$, service rates $a_{i,i+1}(t) = \mu_{i+1}(t)$, and the matrix $A(t)$ has the following structure:

$$A(t) = \begin{pmatrix} a_{00}(t) & \mu_1(t) & 0 & 0 & \cdots & 0 & 0 \\ a_1(t) & a_{11}(t) & \mu_2(t) & 0 & \cdots & 0 & 0 \\ a_2(t) & a_1(t) & a_{22}(t) & \mu_3(t) & \cdots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ a_{N-1}(t) & a_{N-2}(t) & a_{N-3}(t) & a_{N-4}(t) & \cdots & a_{N-1,N-1}(t) & \mu_N(t) \\ a_N(t) & a_{N-1}(t) & a_{N-2}(t) & a_{N-3}(t) & \cdots & a_1(t) & a_{NN}(t) \end{pmatrix}. \tag{2}$$

Here we deal with a model of this class under additional suppositions $a_i(t) = 0,\ 1 \leq i \leq N-1,\ a_N(t) = a(t)$ (only arrival of all customers simultaneously is possible) and $\mu_i(t) \leq \mu_{i+1}(t)$ for any $i, t \geq 0$.

The difficulty of studying this model is due to the fact that it is not possible to apply the most convenient method of the logarithmic norm for it, see [5].

Now we get the following expression for the transposed intensity matrix:

$$A(t) = \begin{pmatrix} -a(t) & \mu_1(t) & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\mu_1(t) & \mu_2(t) & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\mu_2(t) & \mu_3(t) & \cdots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & \cdots & -\mu_{N-1}(t) & \mu_N(t) \\ a(t) & 0 & 0 & 0 & \cdots & 0 & -\mu_N(t) \end{pmatrix}. \tag{3}$$

Put $p_0(t) = 1 - \sum\limits_{i \geq 1} p_i(t)$, then from (1) we obtain

$$\frac{d\mathbf{z}}{dt} = B(t)\,\mathbf{z} + \mathbf{f}(t), \tag{4}$$

where $\mathbf{f}(t) = (0, \ldots, 0, a(t))^T$, $\mathbf{z} = (p_1(t), p_2(t), \ldots, p_N(t))^T$,

$$B(t) = \begin{pmatrix} -\mu_1(t) & \mu_2(t) & 0 & \cdots & 0 & 0 \\ 0 & -\mu_2(t) & \mu_3(t) & \cdots & 0 & 0 \\ 0 & 0 & -\mu_3(t) & \cdots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \cdots & -\mu_{N-1}(t) & \mu_N(t) \\ -a(t) & -a(t) & -a(t) & \cdots & -a(t) & -\mu_N(t) - a(t) \end{pmatrix}. \tag{5}$$

All bounds on the rate of convergence to the limiting regime for $X(t)$ correspond to the same bounds of the solutions of system

$$\frac{d\mathbf{x}}{dt} = B(t)\mathbf{x}(t). \tag{6}$$

Denote by T upper triangular matrix

$$T = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{7}$$

hence

$$T^{-1} = \begin{pmatrix} 1 & -1 & 0 & \ldots & 0 \\ 0 & 1 & -1 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}.$$

Let $\mathbf{u}(t) = T\mathbf{x}(t)$, then

$$\frac{d\mathbf{u}}{dt} = B^*(t)\mathbf{u}(t), \tag{8}$$

where $B^*(t) = TB(t)T^{-1}$, and

$$B^*(t) = \begin{pmatrix} -\mu_1(t) - a(t) & \mu_1(t) & 0 & 0 & \cdots & 0 & 0 \\ -a(t) & -\mu_2(t) & \mu_2(t) & 0 & \cdots & 0 & 0 \\ -a(t) & 0 & -\mu_3(t) & \mu_3(t) & \cdots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ -a(t) & 0 & 0 & 0 & \cdots & -\mu_{N-1}(t) & \mu_{N-1}(t) \\ -a(t) & 0 & 0 & 0 & \cdots & 0 & -\mu_N(t) \end{pmatrix}. \tag{9}$$

Once again, we note that the matrix $B^*(t)$ in is not essentially non-negative, and in such a situation the method of the logarithmic norm is inconvenient to apply (it gives poor results).

For the study of this system, we use the differential inequalities method, which was described in [3, 7].

Let d_i, $i = 1, \ldots, N$ be nonzero numbers, and $\mathsf{D} = diag\,(d_1, d_2, \ldots d_N)$ be a diagonal matrix:

$$\mathsf{D} = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_N \end{pmatrix}. \tag{10}$$

Put $\mathbf{w}(t) = \mathsf{D}\mathbf{u}(t)$, then we obtain from (8) the following system:

$$\frac{d\mathbf{w}}{dt} = B^{**}(t)\mathbf{w}(t), \tag{11}$$

where $B^{**}(t) = \mathsf{D}B^{*}(t)\mathsf{D}^{-1} =$

$$= \begin{pmatrix} -\mu_1(t) - a(t) & \mu_2(t)\cdot\frac{d_1}{d_2} & 0 & 0 & \cdots & 0 & 0 \\ -a(t)\cdot\frac{d_2}{d_1} & -\mu_2(t) & \mu_3(t)\cdot\frac{d_2}{d_3} & 0 & \cdots & 0 & 0 \\ -a(t)\cdot\frac{d_3}{d_1} & 0 & -\mu_3(t) & \mu_4(t)\cdot\frac{d_3}{d_4} & \cdots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ -a(t)\cdot\frac{d_{N-2}}{d_1} & 0 & 0 & 0 & \cdots & -\mu_{N-1}(t) & \mu_N(t)\cdot\frac{d_{N-1}}{d_N} \\ -a(t)\cdot\frac{d_{N-1}}{d_1} & 0 & 0 & 0 & \cdots & 0 & -\mu_N(t) \end{pmatrix}.$$

Let $\mathbf{u}(t)$ be an arbitrary solution of system (8).

Since the function $u_k(t)$ (the $k-$th coordinate of $\mathbf{u}(t)$) is analytic, it has a finite number of zeros on each interval. Consider an interval in which the signs of all the functions $u_k(t)$ do not change, say, (t_1, t_2). Choose the elements of the diagonal matrix such that signs of the entries d_i are equal with signs of corresponding coordinates $u_i(t)$ of the solution of system (8).

Since any $d_k u_k(t) > 0$ on the corresponding time interval, the sum $\sum_{k=1}^{N} d_k u_k = \|\mathbf{w}\|$ can be considered as the corresponding norm.

Moreover, we have the following inequalities:

$$\|\mathbf{u}\| \le N\|\mathbf{x}\|,\ \|\mathbf{x}\| \le 2\|\mathbf{u}\|,\ \|\mathbf{w}\| \le \max_k d_k\|\mathbf{u}\|,\ \|\mathbf{u}\| \le \left(\min_k d_k\right)^{-1}\|\mathbf{w}\|. \tag{12}$$

Let $B^{**}(t) = \left(b_{ij}^{**}(t)\right)_{i,j=1}^{N}$. Now, if the function $\alpha_D(t)$ is such that $\sum_{i=1}^{N} b_{ij}^{**}(t) \le -\alpha_D(t)$, $j = 1, ..., N$, then the following bound holds:

$$\frac{d\,\|\mathbf{w}\|}{dt} = \frac{d\left(\sum_{i=1}^{N} w_i\right)}{dt} = \sum_{j=1}^{N}\sum_{i=1}^{N} b_{ij}^{**}(t) w_j \le -\alpha_D(t)\,\|\mathbf{w}\|,$$

and hence

$$\|\mathbf{w}(t)\| \le e^{-\int_s^t \alpha_D(\tau)d\tau}\|\mathbf{w}(s)\|.$$

Therefore, in the original norm we get the following inequality

$$\|\mathbf{x}(t)\| \le \frac{2N\max_k d_k}{\min_k d_k} e^{-\int_s^t \alpha_D(\tau)d\tau}\|\mathbf{x}(s)\|, \tag{13}$$

for any $t_1 < s \le t < t_2$, and by continuity we get this inequality for $s = t_1$, $t = t_2$. Now we consider all such intervals (there is only a finite number 2^N of intervals with different sign combinations) and put $\alpha^*(t) = \min\{\alpha_D(t)\}$, $C = \max\left(\frac{\max_k d_k}{\min_k d_k}\right)$ where the minimum of $\alpha_D(t)$ and the maximum of $\frac{\max_k d_k}{\min_k d_k}$ is taken over all intervals with different sign combinations of coordinates of the solution. Finally, we obtain the following bound

$$\|\mathbf{x}(t)\| \le 2NCe^{-\int_0^t \alpha^*(\tau)d\tau}\|\mathbf{x}(0)\|. \tag{14}$$

In our case (in general, all intensities depend on the time t)

$$\sum_{i=1}^N w_i' = \left(-\mu_1 - a\cdot\left(1 + \frac{d_2}{d_1} + \frac{d_3}{d_1} + \dots + \frac{d_N}{d_1}\right)\right)\cdot w_1 - \mu_2\cdot\left(1 - \frac{d_1}{d_2}\right)\cdot w_2$$

$$-\mu_3\cdot\left(1 - \frac{d_2}{d_3}\right)\cdot w_3 - \dots - \mu_N\cdot\left(1 - \frac{d_{N-1}}{d_N}\right)\cdot w_N$$

(1) Let all $u_1, ..., u_N$ be positive. Since $\left(1 - \frac{d_i}{d_{i+1}}\right)$ must be positive, we have $d_{i+1} > d_i$. Suppose $d_1 := \varepsilon^N$, $d_2 := \varepsilon^{N-1}$, ..., $d_N := \varepsilon$, then

$$\sum_{i=1}^N w_i' = \left(-\mu_1 - a\cdot\left(1 + \frac{d_2}{d_1} + \frac{d_3}{d_1} + \dots + \frac{d_N}{d_1}\right)\right)\cdot w_1 - \mu_2\cdot\left(1 - \frac{d_1}{d_2}\right)\cdot w_2$$

$$-\mu_3\cdot\left(1 - \frac{d_2}{d_3}\right)\cdot w_3 - \dots - \mu_N\cdot\left(1 - \frac{d_{N-1}}{d_N}\right)\cdot w_N$$

$$= \left(-\mu_1 - a\cdot\left(1 + \frac{1}{\varepsilon} + \frac{1}{\varepsilon^2} + \dots + \frac{1}{\varepsilon^{N-1}}\right)\right)\cdot w_1 - \mu_2\cdot(1-\varepsilon)\cdot w_2$$

$$-\mu_3\cdot(1-\varepsilon)\cdot w_3 - \dots - \mu_N\cdot(1-\varepsilon)\cdot w_N,$$

and we have for the corresponding interval $\alpha_D = \min\{\mu_i\cdot(1-\varepsilon)\} = \mu_1\cdot(1-\varepsilon)$.

(2) Let all $u_1, ..., u_k$ be positive, and all $u_{k+1}, ..., u_N$ negative. Similarly $|d_{i+1}| > |d_i|$. Suppose $d_1 := \varepsilon^k$, $d_2 := \varepsilon^{k-1}$, ..., $d_k := \varepsilon$, $d_{k+1} := -\varepsilon^N$, $d_{k+2} := -\varepsilon^{N-1}$, ... , $d_N := -\varepsilon^{k+1}$, then

$$\sum_{i=1}^{N} w_i' = \left(-\mu_1 - a \cdot \left(1 + \frac{d_2}{d_1} + \frac{d_3}{d_1} + \cdots + \frac{d_N}{d_1}\right)\right) \cdot w_1 - \mu_2 \cdot \left(1 - \frac{d_1}{d_2}\right) \cdot w_2$$

$$-\mu_3 \cdot \left(1 - \frac{d_2}{d_3}\right) \cdot w_3 - \cdots - \mu_N \cdot \left(1 - \frac{d_{N-1}}{d_N}\right) \cdot w_N$$

$$= \left(-\mu_1 - a \cdot \left(1 + \frac{1}{\varepsilon} + \frac{1}{\varepsilon^2} + \cdots + \frac{1}{\varepsilon^{k-1}} - \varepsilon^{N-k} - \varepsilon^{N-k-1} - \ldots - \varepsilon\right)\right) \cdot w_1$$

$$-\mu_2 \cdot (1-\varepsilon) \cdot w_2 - \mu_3 \cdot (1-\varepsilon) \cdot w_3 - \cdots - \mu_k \cdot (1-\varepsilon) \cdot w_k - \mu_{k+1} \cdot \left(1 + \frac{1}{\varepsilon^{N-1}}\right) \cdot w_{k+1}$$

$$-\mu_{k+2} \cdot (1-\varepsilon) \cdot w_{k+2} - \ldots - \mu_N \cdot (1-\varepsilon) \cdot w_N.$$

In this case we also have the corresponding interval $\alpha_D = \min\{\mu_i \cdot (1-\varepsilon)\} = \mu_1 \cdot (1-\varepsilon)$.

Every time we changing sign on going from u_s to u_{s+1} we suppose $|d_{s+1}|$ be equal ε^m, where m is the number of the last element period of consistency.

Then we have $C = \varepsilon^{1-N}$, and the following bounds hold:

$$\|x(t)\| \le 2N\varepsilon^{1-N} e^{-\mu_1 \cdot (1-\varepsilon) t} \|x(0)\|, \tag{15}$$

for the homogeneous Markov chain (constant intensities);
and

$$\|x(t)\| \le 2N\varepsilon^{1-N} e^{-(1-\varepsilon)\int_0^t \mu_1(\tau)d\tau} \|x(0)\|, \tag{16}$$

in general situation.

2 Example

Consider here a specific queueing model with 1−periodic intensities. Let $a(t) = \lambda(t) = 2 + \sin(2\pi t)$ and $\mu_k(t) = k\,(2 + \cos(2\pi t))$. Then $A(t) =$

$$= \begin{pmatrix} -(2+\sin(2\pi t)) & 2+\cos(2\pi t) & 0 & \cdots & 0 & 0 \\ 0 & -(2+\cos(2\pi t)) & 2\cdot(2+\cos(2\pi t)) & \cdots & 0 & 0 \\ 0 & 0 & -2\cdot(2+\cos(2\pi t)) & \cdots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \cdots & -(N-1)(2+\cos(2\pi t)) & N\cdot(2+\cos(2\pi t)) \\ 2+\sin(2\pi t) & 0 & 0 & \cdots & 0 & -N\cdot(2+\cos(2\pi t)) \end{pmatrix},$$

$B^{**}(t) =$

$$\begin{pmatrix} -(2+\cos(2\pi t))-(2+\sin(2\pi t)) & 2\cdot(2+\cos(2\pi t))\cdot\frac{d_1}{d_2} & 0 & \cdots & 0 \\ -(2+\sin(2\pi t))\cdot\frac{d_2}{d_1} & -2\cdot(2+\cos(2\pi t)) & 3\cdot(2+\cos(2\pi t))\cdot\frac{d_2}{d_3} & \cdots & 0 \\ -(2+\sin(2\pi t))\cdot\frac{d_3}{d_1} & 0 & -3\cdot(2+\cos(2\pi t)) & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ -(2+\sin(2\pi t))\cdot\frac{d_{N-1}}{d_1} & 0 & 0 & \cdots & N\cdot(2+\cos(2\pi t))\cdot\frac{d_{N-1}}{d_N} \\ -(2+\sin(2\pi t))\cdot\frac{d_N}{d_1} & 0 & 0 & \cdots & -N\cdot(2+\cos(2\pi t)) \end{pmatrix},$$

and we have

$$\sum_{i=1}^{N} w_i' = (-(2+\cos(2\pi t)) - (2+\sin(2\pi t))\cdot(1+\frac{d_2}{d_1}+\frac{d_3}{d_1}+\cdots+\frac{d_N}{d_1}))\cdot w_1$$

$$-2\cdot(2+\cos(2\pi t))\cdot(1-\frac{d_1}{d_2})\cdot w_2 - 3\cdot(2+\cos(2\pi t))\cdot(1-\frac{d_2}{d_3})\cdot w_3$$

$$-\cdots - N\cdot(2+\cos(2\pi t))\cdot(1-\frac{d_{N-1}}{d_N})\cdot w_N.$$

Then we have the following bound on the rate of convergence

$$\|\mathbf{x}(t)\| \le 2N\varepsilon^{1-N}e^{-(1-\varepsilon)t}\|\mathbf{x}(0)\|. \tag{17}$$

Let $N = 200$. Then for any $\varepsilon \in (0, 1)$, we obtain the corresponding bound on the rate of convergence.

Denote by $E(t, k) = E(X(t)|X(0) = k)$ the conditional expected number of customers in the queue at instant t, provided that initially (at instant $t = 0$) k customers were present in the queue.

We compute here the probability of the empty queue $p_0(t)$ and the mathematical expectation of the number of customers in the queue $E(t, k)$, as it is shown on the Figs. 1, 2, 3 and 4.

These graphs are obtained using our standard approach (see detailed description in [4]) for solving numerically the forward Kolmogorov system on the corresponding interval and find approximately the limiting characteristics of this queueing model.

Note, that the bound (17) guarantees the coincidence of the probability characteristics for the queue-length process with different initial conditions with a predetermined accuracy for the corresponding (sufficiently large) values of t. In fact, as the graphs show, the difference is already quite small at $t \ge 17$.

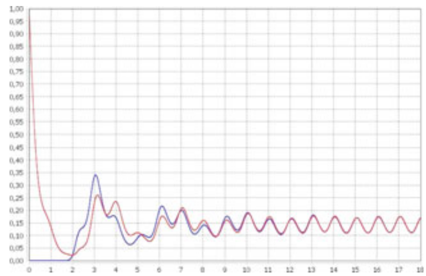

Fig. 1 Example. Probability of the empty queue for $t \in [0, 18]$ with initial conditions $X(0) = 0$ (red) and $X(0) = 200$ (blue)

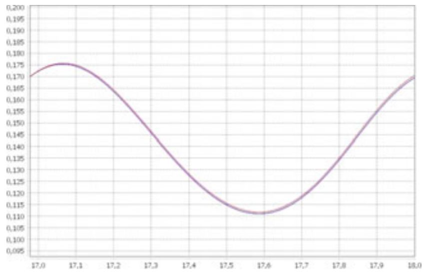

Fig. 2 Example. Probability of the empty queue for $t \in [17, 18]$ with initial conditions $X(0) = 0$ and $X(0) = 200$

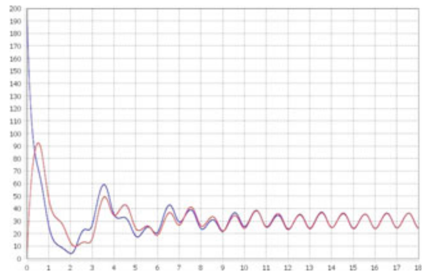

Fig. 3 Example. The mean $E(t, k)$ for $t \in [0, 18]$ with initial conditions $X(0) = 0$ (red) and $X(0) = 200$ (blue)

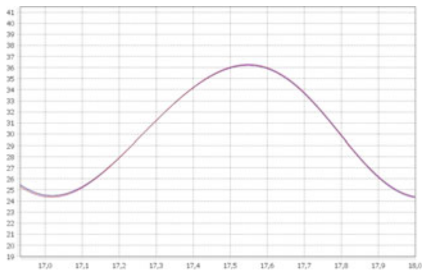

Fig. 4 Example. The mean $E(t, k)$ for $t \in [17, 18]$ with initial conditions $X(0) = 0$ and $X(0) = 200$

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References

1. Li, J., Zhang, L.: M X/M/c queue with catastrophes and state-dependent control at idle time. Front. Math. China **12**(6), 1427–1439 (2017)
2. Nelson, R., Towsley, D., Tantawi, A.N.: Performance analysis of parallel processing systems. IEEE Trans. Softw. Eng. **14**(4), 532–540 (1988)
3. Satin, Y., Zeifman, A., Kryukova, A.: On the rate of convergence and limiting characteristics for a nonstationary queueing model. Mathematics **7**(8), 678 (2019)
4. Zeifman, A., Satin, Y., Korolev, V., Shorgin, S.: On truncations for weakly ergodic inhomogeneous birth and death processes. Int. J. Appl. Math. Comput. Sci. **24**, 503–518 (2014)
5. Zeifman, A., Razumchik, R., Satin, Y., Kiseleva, K., Korotysheva, A., Korolev, V.: Bounds on the rate of convergence for one class of inhomogeneous Markovian queueing models with possible batch arrivals and services. Int. J. Appl. Math. Comput. Sci. **28**, 141–154 (2018)
6. Zeifman, A., Sipin, A., Korolev, V., Shilova, G., Kiseleva, K., Korotysheva, A., Satin, Y.: On sharp bounds on the rate of convergence for finite continuous-time Markovian queueing models. LNCS, vol. 10672, pp. 20–28 (2018)
7. Zeifman, A., Satin, Y., Kiseleva, K., Kryukova, A.: Applications of differential inequalities to bounding the rate of convergence for continuous-time Markov chains. In: AIP Conference Proceedings, vol. 2116, p. 090009 (2019)

Application of method of differential inequalities to bounding the rate of convergence for a class of Markov chains

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Abstract We consider the linear system of differential equations $\frac{d\mathbf{p}}{dt} = A(t)\mathbf{p}$, which is the forward Kolmogorov system, for a class of Markov chains with 'batch' births and single deaths. We apply the method of differential inequalities for obtaining bounds on the rate of convergence for the system. A specific queueing model is considered and the corresponding limiting characteristics are computing.

1 Introduction and general bounds

Let $\{X(t),\ t \geq 0\}$ be a continuous-time Markov chain with finite state space $\mathscr{X} = \{0, 1, \ldots, N\}$. Denote by $p_{ij}(s,t) = P\{X(t) = j\,|X(s) = i\}$, $i, j \geq 0$, $0 \leq s \leq t$ the transition probabilities of $X(t)$ and by $p_i(t) = P\{X(t) = i\}$ – the probability that the Markov chain $X(t)$ is in state i at time t. Let $\mathbf{p}(t) = (p_0(t), p_1(t), \ldots)^T$ be the vector of state probabilities at the moment t.

Then the probabilistic dynamics of the process $\{X(t),\ t \geq 0\}$ is described by the forward Kolmogorov system

$$\frac{d\mathbf{p}}{dt} = A(t)\mathbf{p}, \tag{1}$$

where $A(t) = Q^T(t)$ is the transposed intensity matrix. All column sums of this matrix are zeros for any $t \geq 0$, and $A(t)$ is essentially nonnegative (i.e. all its off-diagonal elements are nonnegative for any $t \geq 0$).

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We suppose that all 'intensity functions' $a_{ij}(t)$ are analytic in t for $t \geq 0$.

Consider a queueing model for a queue with batch arrivals and single services, see the first motivation in [2] and more recent studies in [1, 5, 6].

Then we have $a_{ij}(t) = 0$ for $i < j-1$, all arrival rates do not depend on the size of a queue, i.e. $a_{i+k,i}(t) = a_k(t)$ for $k \geq 1$, service rates $a_{i,i+1}(t) = \mu_{i+1}(t)$, and the matrix $A(t)$ has the following structure:

$$A(t) = \begin{pmatrix} a_{00}(t) & \mu_1(t) & 0 & 0 & \cdots & 0 & 0 \\ a_1(t) & a_{11}(t) & \mu_2(t) & 0 & \cdots & 0 & 0 \\ a_2(t) & a_1(t) & a_{22}(t) & \mu_3(t) & \cdots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ a_{N-1}(t) & a_{N-2}(t) & a_{N-3}(t) & a_{N-4}(t) & \cdots & a_{N-1,N-1}(t) & \mu_N(t) \\ a_N(t) & a_{N-1}(t) & a_{N-2}(t) & a_{N-3}(t) & \cdots & a_1(t) & a_{NN}(t) \end{pmatrix}. \tag{2}$$

Here we deal with a model of this class under additional suppositions $a_i(t) = 0$, $1 \leq i \leq N-1$, $a_N(t) = a(t)$ (only arrival of all customers simultaneously is possible) and $\mu_i(t) \leq \mu_{i+1}(t)$ for any i, $t \geq 0$.

The difficulty of studying this model is due to the fact that it is not possible to apply the most convenient method of the logarithmic norm for it, see [5].

Now we get the following expression for the transposed intensity matrix:

$$A(t) = \begin{pmatrix} -a(t) & \mu_1(t) & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\mu_1(t) & \mu_2(t) & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\mu_2(t) & \mu_3(t) & \cdots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & \cdots & -\mu_{N-1}(t) & \mu_N(t) \\ a(t) & 0 & 0 & 0 & \cdots & 0 & -\mu_N(t) \end{pmatrix}. \tag{3}$$

Put $p_0(t) = 1 - \sum\limits_{i \geq 1} p_i(t)$, then from (1) we obtain

$$\frac{d\mathbf{z}}{dt} = B(t)\mathbf{z} + \mathbf{f}(t), \tag{4}$$

where $\mathbf{f}(t) = (0, \ldots, 0, a(t))^T$, $\mathbf{z} = (p_1(t), p_2(t), \ldots, p_N(t))^T$,

$$B(t) = \begin{pmatrix} -\mu_1(t) & \mu_2(t) & 0 & \cdots & 0 & 0 \\ 0 & -\mu_2(t) & \mu_3(t) & \cdots & 0 & 0 \\ 0 & 0 & -\mu_3(t) & \cdots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \cdots & -\mu_{N-1}(t) & \mu_N(t) \\ -a(t) & -a(t) & -a(t) & \cdots & -a(t) & -\mu_N(t) - a(t) \end{pmatrix}. \tag{5}$$

All bounds on the rate of convergence to the limiting regime for $X(t)$ correspond to the same bounds of the solutions of system

$$\frac{d\mathbf{x}}{dt} = B(t)\mathbf{x}(t). \tag{6}$$

Denote by T upper triangular matrix

$$T = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{7}$$

hence

$$T^{-1} = \begin{pmatrix} 1 & -1 & 0 & \ldots & 0 \\ 0 & 1 & -1 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}.$$

Let $\mathbf{u}(t) = T\mathbf{x}(t)$, then

$$\frac{d\mathbf{u}}{dt} = B^*(t)\mathbf{u}(t), \tag{8}$$

where $B^*(t) = TB(t)T^{-1}$, and

$$B^*(t) = \begin{pmatrix} -\mu_1(t)-a(t) & \mu_1(t) & 0 & 0 & \cdots & 0 & 0 \\ -a(t) & -\mu_2(t) & \mu_2(t) & 0 & \cdots & 0 & 0 \\ -a(t) & 0 & -\mu_3(t) & \mu_3(t) & \cdots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ -a(t) & 0 & 0 & 0 & \cdots & -\mu_{N-1}(t) & \mu_{N-1}(t) \\ -a(t) & 0 & 0 & 0 & \cdots & 0 & -\mu_N(t) \end{pmatrix}. \tag{9}$$

Once again, we note that the matrix $B^*(t)$ in is not essentially non-negative, and in such a situation the method of the logarithmic norm is inconvenient to apply (it gives poor results).

For the study of this system, we use the differential inequalities method, which was described in [3, 7].

Let d_i, $i = 1,\ldots,N$ be nonzero numbers, and $\mathsf{D} = diag\,(d_1, d_2, \ldots d_N)$ be a diagonal matrix:

$$\mathsf{D} = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_N \end{pmatrix}. \tag{10}$$

Put $\mathbf{w}(t) = \mathsf{D}\mathbf{u}(t)$, then we obtain from (8) the the following system:

$$\frac{d\mathbf{w}}{dt} = B^{**}(t)\mathbf{w}(t), \tag{11}$$

where $B^{**}(t) = \mathsf{D}B^{*}(t)\mathsf{D}^{-1} =$

$$= \begin{pmatrix} -\mu_1(t)-a(t) & \mu_2(t)\cdot\frac{d_1}{d_2} & 0 & 0 & \cdots & 0 & 0 \\ -a(t)\cdot\frac{d_2}{d_1} & -\mu_2(t) & \mu_3(t)\cdot\frac{d_2}{d_3} & 0 & \cdots & 0 & 0 \\ -a(t)\cdot\frac{d_3}{d_1} & 0 & -\mu_3(t) & \mu_4(t)\cdot\frac{d_3}{d_4} & \cdots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ -a(t)\cdot\frac{d_{N-2}}{d_1} & 0 & 0 & 0 & \cdots & -\mu_{N-1}(t) & \mu_N(t)\cdot\frac{d_{N-1}}{d_N} \\ -a(t)\cdot\frac{d_{N-1}}{d_1} & 0 & 0 & 0 & \cdots & 0 & -\mu_N(t) \end{pmatrix}.$$

Let $\mathbf{u}(t)$ be an arbitrary solution of system (8).

Since the function $u_k(t)$ (the $k-$th coordinate of $\mathbf{u}(t)$) is analytic, it has a finite number of zeros on each interval. Consider an interval in which the signs of all the functions $u_k(t)$ do not change, say, (t_1,t_2). Choose the elements of the diagonal matrix such that signs of the entries d_i are equal with signs of corresponding coordinates $u_i(t)$ of the solution of system (8).

Since any $d_k u_k(t) > 0$ on the corresponding time interval, the sum $\sum_{k=1}^{N} d_k u_k = \|\mathbf{w}\|$ can be considered as the corresponding norm.

Moreover, we have the following inequalities:

$$\|\mathbf{u}\| \le N\|\mathbf{x}\|, \|\mathbf{x}\| \le 2\|\mathbf{u}\|, \|\mathbf{w}\| \le \max_k d_k\|\mathbf{u}\|, \|\mathbf{u}\| \le \left(\min_k d_k\right)^{-1}\|\mathbf{w}\|. \tag{12}$$

Let $B^{**}(t) = \left(b_{ij}^{**}(t)\right)_{i,j=1}^{N}$. Now, if the function $\alpha_D(t)$ is such that $\sum_{i=1}^{N} b_{ij}^{**}(t) \le -\alpha_D(t)$, $j = 1,...,N$, then the following bound holds:

$$\frac{d\,\|\mathbf{w}\|}{dt} = \frac{d\left(\sum_{i=1}^{N} w_i\right)}{dt} = \sum_{j=1}^{N}\sum_{i=1}^{N} b_{ij}^{**}(t)w_j \le -\alpha_D(t)\,\|\mathbf{w}\|\,,$$

and hence

$$\|\mathbf{w}(t)\| \le e^{-\int_s^t \alpha_D(\tau)d\tau}\|\mathbf{w}(s)\|.$$

Therefore, in the original norm we get the following inequality

$$\|\mathbf{x}(t)\| \le \frac{2N\max_k d_k}{\min_k d_k} e^{-\int_s^t \alpha_D(\tau)d\tau}\|\mathbf{x}(s)\|, \tag{13}$$

for any $t_1 < s \le t < t_2$, and by continuity we get this inequality for $s = t_1$, $t = t_2$. Now we consider all such intervals (there is only a finite number 2^N of intervals with different sign combinations) and put $\alpha^*(t) = \min\{\alpha_D(t)\}$, $C = \max\left(\frac{\max_k d_k}{\min_k d_k}\right)$ where the minimum of $\alpha_D(t)$ and the maximum of $\frac{\max_k d_k}{\min_k d_k}$ is taken over all intervals with different sign combinations of coordinates of the solution. Finally, we obtain the following bound

$$\|\mathbf{x}(t)\| \le 2NCe^{-\int_0^t \alpha^*(\tau)d\tau}\|\mathbf{x}(0)\|. \tag{14}$$

In our case (in general, all intensities depend on the time t)

$$\sum_{i=1}^{N} w_i' = \left(-\mu_1 - a\cdot\left(1+\frac{d_2}{d_1}+\frac{d_3}{d_1}+\cdots+\frac{d_N}{d_1}\right)\right)\cdot w_1 - \mu_2\cdot\left(1-\frac{d_1}{d_2}\right)\cdot w_2 -$$

$$-\mu_3\cdot\left(1-\frac{d_2}{d_3}\right)\cdot w_3 - \cdots - \mu_N\cdot\left(1-\frac{d_{N-1}}{d_N}\right)\cdot w_N$$

1) Let all u_1, ..., u_N be positive. Since $\left(1-\frac{d_i}{d_{i+1}}\right)$ must be positive, we have $d_{i+1} > d_i$. Suppose $d_1 := \varepsilon^N$, $d_2 := \varepsilon^{N-1}$, ..., $d_N := \varepsilon$, then

$$\sum_{i=1}^{N} w_i' = \left(-\mu_1 - a\cdot\left(1+\frac{d_2}{d_1}+\frac{d_3}{d_1}+\cdots+\frac{d_N}{d_1}\right)\right)\cdot w_1 - \mu_2\cdot\left(1-\frac{d_1}{d_2}\right)\cdot w_2 -$$

$$-\mu_3\cdot\left(1-\frac{d_2}{d_3}\right)\cdot w_3 - \cdots - \mu_N\cdot\left(1-\frac{d_{N-1}}{d_N}\right)\cdot w_N =$$

$$= \left(-\mu_1 - a\cdot\left(1+\frac{1}{\varepsilon}+\frac{1}{\varepsilon^2}+\cdots+\frac{1}{\varepsilon^{N-1}}\right)\right)\cdot w_1 - \mu_2\cdot(1-\varepsilon)\cdot w_2 -$$

$$-\mu_3\cdot(1-\varepsilon)\cdot w_3 - \cdots - \mu_N\cdot(1-\varepsilon)\cdot w_N,$$

and we have for the corresponding interval $\alpha_D = \min\{\mu_i\cdot(1-\varepsilon)\} = \mu_1\cdot(1-\varepsilon)$.

2) Let all u_1, ..., u_k be positive, and all u_{k+1}, ..., u_N negative. Similarly $|d_{i+1}| > |d_i|$. Suppose $d_1 := \varepsilon^k$, $d_2 := \varepsilon^{k-1}$, ..., $d_k := \varepsilon$, $d_{k+1} := -\varepsilon^N$, $d_{k+2} := -\varepsilon^{N-1}$, ... , $d_N := -\varepsilon^{k+1}$, then

$$\sum_{i=1}^{N} w_i' = \left(-\mu_1 - a\cdot\left(1+\frac{d_2}{d_1}+\frac{d_3}{d_1}+\cdots+\frac{d_N}{d_1}\right)\right)\cdot w_1 - \mu_2\cdot\left(1-\frac{d_1}{d_2}\right)\cdot w_2 -$$

$$-\mu_3\cdot\left(1-\frac{d_2}{d_3}\right)\cdot w_3 - \cdots - \mu_N\cdot\left(1-\frac{d_{N-1}}{d_N}\right)\cdot w_N =$$

$$= \left(-\mu_1 - a\cdot\left(1+\frac{1}{\varepsilon}+\frac{1}{\varepsilon^2}+\cdots+\frac{1}{\varepsilon^{k-1}}-\varepsilon^{N-k}-\varepsilon^{N-k-1}-\ldots-\varepsilon\right)\right)\cdot w_1 -$$

$$-\mu_2\cdot(1-\varepsilon)\cdot w_2 - \mu_3\cdot(1-\varepsilon)\cdot w_3 - \cdots - \mu_k\cdot(1-\varepsilon)\cdot w_k - \mu_{k+1}\cdot\left(1+\frac{1}{\varepsilon^{N-1}}\right)\cdot w_{k+1} -$$

$$-\mu_{k+2}\cdot(1-\varepsilon)\cdot w_{k+2} - \ldots - \mu_N\cdot(1-\varepsilon)\cdot w_N.$$

In this case we also have the the corresponding interval $\alpha_D = \min\{\mu_i \cdot (1-\varepsilon)\} = \mu_1 \cdot (1-\varepsilon)$.

Every time we changing sign on going from u_s to u_{s+1} we suppose $|d_{s+1}|$ be equal ε^m, where m is the number of the last element period of consistency.

Then we have $C = \varepsilon^{1-N}$, and the following bounds hold:

$$\|x(t)\| \leq 2N\varepsilon^{1-N} e^{-\mu_1\cdot(1-\varepsilon)t} \|x(0)\|, \tag{15}$$

for the homogeneous Markov chain (constant intensities);
and

$$\|x(t)\| \leq 2N\varepsilon^{1-N} e^{-(1-\varepsilon)\int_0^t \mu_1(\tau)d\tau} \|x(0)\|, \tag{16}$$

in general situation.

2 Example

Consider here a specific queueing model with 1−periodic intensities. Let $a(t) = \lambda(t) = 2+\sin(2\pi t)$ and $\mu_k(t) = k\,(2+\cos(2\pi t))$. Then $A(t) =$

$$= \begin{pmatrix} -(2+\sin(2\pi t)) & 2+\cos(2\pi t) & 0 & \cdots & 0 & 0 \\ 0 & -(2+\cos(2\pi t)) & 2\cdot(2+\cos(2\pi t)) & \cdots & 0 & 0 \\ 0 & 0 & -2\cdot(2+\cos(2\pi t)) & \cdots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \cdots & -(N-1)(2+\cos(2\pi t)) & N\cdot(2+\cos(2\pi t)) \\ 2+\sin(2\pi t) & 0 & 0 & \cdots & 0 & -N\cdot(2+\cos(2\pi t)) \end{pmatrix},$$

$B^{**}(t) =$

$$\begin{pmatrix} -(2+\cos(2\pi t))-(2+\sin(2\pi t)) & 2\cdot(2+\cos(2\pi t))\cdot\frac{d_1}{d_2} & 0 & \cdots & 0 \\ -(2+\sin(2\pi t))\cdot\frac{d_2}{d_1} & -2\cdot(2+\cos(2\pi t)) & 3\cdot(2+\cos(2\pi t))\cdot\frac{d_2}{d_3} & \cdots & 0 \\ -(2+\sin(2\pi t))\cdot\frac{d_3}{d_1} & 0 & -3\cdot(2+\cos(2\pi t)) & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ -(2+\sin(2\pi t))\cdot\frac{d_{N-1}}{d_1} & 0 & 0 & \cdots & N\cdot(2+\cos(2\pi t))\cdot\frac{d_{N-1}}{d_N} \\ -(2+\sin(2\pi t))\cdot\frac{d_N}{d_1} & 0 & 0 & \cdots & -N\cdot(2+\cos(2\pi t)) \end{pmatrix},$$

and we have

$$\sum_{i=1}^{N} w_i' = (-(2+\cos(2\pi t)) - (2+\sin(2\pi t))\cdot(1+\frac{d_2}{d_1}+\frac{d_3}{d_1}+\cdots+\frac{d_N}{d_1}))\cdot w_1$$

$$-2\cdot(2+\cos(2\pi t))\cdot(1-\frac{d_1}{d_2})\cdot w_2 - 3\cdot(2+\cos(2\pi t))\cdot(1-\frac{d_2}{d_3})\cdot w_3 - \cdots -$$

$$-N\cdot(2+\cos(2\pi t))\cdot(1-\frac{d_{N-1}}{d_N})\cdot w_N.$$

Then we have the following bound on the rate of convergence

$$\|\mathbf{x}(t)\| \leq 2N\varepsilon^{1-N} e^{-(1-\varepsilon)t} \|\mathbf{x}(0)\|. \tag{17}$$

Let $N = 200$. Then for any $\varepsilon \in (0,1)$, we obtain the corresponding bound on the rate of convergence.

Denote by $E(t,k) = E(X(t)|X(0) = k)$ the conditional expected number of customers in the queue at instant t, provided that initially (at instant $t = 0$) k customers were present in the queue.

We compute here the probability of the empty queue $p_0(t)$ and the mathematical expectation of the number of customers in the queue $E(t,k)$, as it is shown on the Pictures 1–4.

These graphs are obtained using our standard approach (see detailed description in [4]) for solving numerically the forward Kolmogorov system on the corresponding interval and find approximately the limiting characteristics of this queueing model.

Note, that the bound (17) guarantees the coincidence of the probability characteristics for the queue-length process with different initial conditions with a predetermined accuracy for the corresponding (sufficiently large) values of t. In fact, as the graphs show, the difference is already quite small at $t \geq 17$.

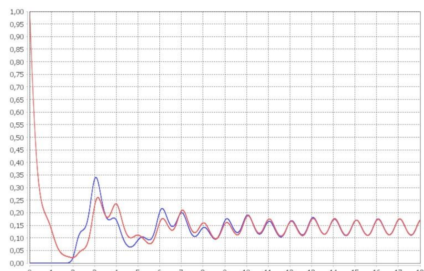

Fig. 1 Example. Probability of the empty queue for $t \in [0,18]$ with initial conditions $X(0) = 0$ (red) and $X(0) = 200$ (blue).

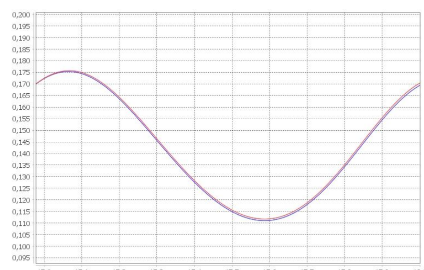

Fig. 2 Example. Probability of the empty queue for $t \in [17,18]$ with initial conditions $X(0) = 0$ and $X(0) = 200$.

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References

1. Li, J., Zhang, L.: M X/M/c Queue with catastrophes and state-dependent control at idle time. Frontiers of Mathematics in China, **12**(6), 1427–1439 (2017)
2. Nelson, R., Towsley, D., Tantawi, A. N.: Performance analysis of parallel processing systems. The IEEE Transactions on Software Engineering, **14**(4), 532–540 (1988)

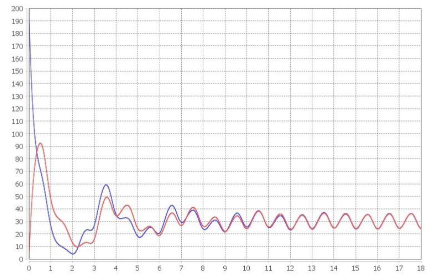

Fig. 3 Example. The mean $E(t,k)$ for $t \in [0,18]$ with initial conditions $X(0) = 0$ (red) and $X(0) = 200$ (blue).

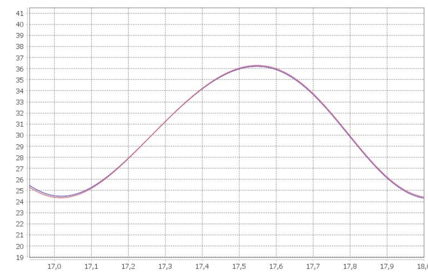

Fig. 4 Example. The mean $E(t,k)$ for $t \in [17,18]$ with initial conditions $X(0) = 0$ and $X(0) = 200$.

3. Satin, Y., Zeifman, A., Kryukova, A.: On the Rate of Convergence and Limiting Characteristics for a Nonstationary Queueing Model. Mathematics, **7**(8), 678 (2019)
4. Zeifman, A., Satin, Ya., Korolev, V., Shorgin, S.: On truncations for weakly ergodic inhomogeneous birth and death processes. Int. J. Appl. Math. Comp. Sci., **24**, 503–518 (2014)
5. Zeifman, A., Razumchik, R., Satin, Y., Kiseleva, K., Korotysheva, A., Korolev, V.: Bounds on the rate of convergence for one class of inhomogeneous Markovian queueing models with possible batch arrivals and services. International Journal of Applied Mathematics and Computer Science, **28**, 141–154 (2018)
6. Zeifman, A., Sipin, A., Korolev, V., Shilova, G., Kiseleva, K., Korotysheva, A., Satin, Y.: On Sharp Bounds on the Rate of Convergence for Finite Continuous-time Markovian Queueing Models, LNCS 2018, **10672**, 20–28 (2018)
7. Zeifman, A., Satin, Y., Kiseleva, K., Kryukova, A.: Applications of Differential Inequalities to Bounding the Rate of Convergence for Continuous-time Markov Chains. AIP Conf. Proc. **2116**, 090009 (2019)