Statistics and Probability Letters 176 (2021) 109150

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Bounds on the rate of convergence for Markovian queuing models with catastrophes

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ARTICLE INFO

Article history:
Received 27 November 2020
Received in revised form 4 April 2021
Accepted 3 May 2021
Available online 18 May 2021

Keywords:
Continuous-time Markov chains
Catastrophes
Queueing models
Rate of convergence

ABSTRACT

In this note, a general approach to the study of non-stationary Markov chains with catastrophes and the corresponding queuing models is considered, as well as to obtain estimates of the limiting regime itself. As an illustration, an example of a queuing model is studied.

1. Introduction

We consider a general nonstationary Markovian queuing model under additional assumption of possibility of catastrophes of the system. As a rule this assumption is sufficient for ergodicity of the corresponding queue-length process.

There are a number of investigations in this area, see, for instance, Ammar (2014), Ammar et al. (2021), Chakravarthy (2017), Chen and Renshaw (1997, 2004), Di Crescenzo et al. (2008), Dudin and Karolik (2001), Li and Zhang (2017), Zhang and Li (2015), Zeifman and Korotysheva (2012), Zeifman et al. (2017b,a, 2020) and references therein.

In these papers as a rule stationary distributions or transient behavior are studied. In our previous papers we obtained estimates on the rate of convergence to the limiting regime for a number of classes of Markovian queuing models with catastrophes. This note is devoted to a simple and general method for study of ergodicity of such models (in particular, in nonstationary situations). This approach enables us to more efficiently compute the main probabilistic characteristics for Markovian queuing models, as shown in Zeifman et al. (2021b).

Here we obtain upper bounds on the rate of convergence for such models and apply these estimates to some specific situations.

Let $\lambda_{i,i+k}(t)$ be the intensity of arrival of group of k customers to the queue at the moment t, if the current length of queue equals i;

$\mu_{i,i-k}(t)$ be the intensity of service of a group of k customers to the queue at the moment t, if the current length of queue equals i.

In addition, we separately introduce a special notation for the catastrophe (disaster) intensity, that is, the intensity of simultaneous loss of all customers. Namely, let $\beta_k(t)$ be a disaster (catastrophe) intensity, if the current size of the length of queue equals k.

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https://doi.org/10.1016/j.spl.2021.109150

Consider the corresponding queue-length process $X(t)$. Then the intensity matrix $Q(t) = \left(q_{ij}(t)\right)_{i,j=0}^{\infty}$ for $X(t)$ takes the following form:

$$Q\left(t\right)=\begin{pmatrix} q_{00}\left(t\right) & \lambda_{01}\left(t\right) & \lambda_{02}\left(t\right) & \lambda_{03}\left(t\right) & \lambda_{04}\left(t\right) & \cdots & \cdots \\ \mu_{10}(t)+\beta_1\left(t\right) & q_{11}\left(t\right) & \lambda_{12}\left(t\right) & \lambda_{13}\left(t\right) & \cdots & \cdots & \cdots \\ \mu_{20}(t)+\beta_2\left(t\right) & \mu_{21}(t)\left(t\right) & q_{22}\left(t\right) & \lambda_{23}(t)\left(t\right) & \lambda_{24}\left(t\right) & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mu_{j0}(t)+\beta_j\left(t\right) & & \cdots & \mu_{j,j-1}\left(t\right) & q_{jj}\left(t\right) & \lambda_{j,j+1}\left(t\right) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $q_{ii}\left(t\right)$'s are such that all row sums of the matrix are equal to zero for any $t \geq 0$.

Applying the standard approach (see for instance Zeifman et al., 2017b; Zeifman, 2020) we assume that all the intensity functions $q_{ij}(t)$ are locally integrable on $[0, \infty)$, and that $\sup_i |q_{ii}(t)| = L(t) < \infty$, for almost all $t \geq 0$.

Then the probabilistic dynamics of the process $\{X(t),\ t \geq 0\}$ is given by the forward Kolmogorov system

$$\frac{d\mathbf{p}(t)}{dt} = A(t)\mathbf{p}(t), \tag{1}$$

where

$$A(t) = Q^T(t) = \begin{pmatrix} q_{00}\left(t\right) & \mu_{10}\left(t\right)+\beta_1\left(t\right) & \mu_{20}\left(t\right)+\beta_2\left(t\right) & \cdots & \mu_{j0}\left(t\right)+\beta_j\left(t\right) & \cdots & \\ \lambda_{01}\left(t\right) & q_{11}\left(t\right) & \mu_{21}\left(t\right) & \mu_{31}\left(t\right) & \cdots & \cdots & \cdots \\ \lambda_{02}\left(t\right) & \lambda_{12}\left(t\right) & q_{22}\left(t\right) & \mu_{32}\left(t\right) & \mu_{42}\left(t\right) & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is the transposed intensity matrix and $\mathbf{p}(t)$ is the column vector of state probabilities, $\mathbf{p}(t) = (p_0(t), p_1(t), \ldots)^T$.

Then, applying the modified combined approach of Zeifman et al. (2017a, 2018) we can obtain bounds on the rate of convergence of the queue-length process to its limiting characteristics and compute them. We separately consider the important special cases.

2. Basic notions

Denote by $p_{ij}(s,t) = P\{X(t) = j\,|X(s) = i\},\ i,j \geq 0,\ \ 0 \leq s \leq t$ the transition probabilities of $X(t)$ and by $p_i(t) = P\{X(t) = i\}$ the probability that $X(t)$ is in state i at time t. Let $\mathbf{p}(t) = (p_0(t), p_1(t), \ldots)^T$ be probability distribution vector at instant t.

Throughout the paper by $\|\cdot\|$ we denote the l_1-norm, i. e. $\|\mathbf{p}(t)\| = \sum_k size : p_k(t) size :$, and $\|A(t)\| = \sup_j \sum_i size : a_{ij}(t) size :$. Let Ω be a set of all stochastic vectors, i.e. l_1 vectors with non-negative coordinates and unit norm. Hence we have $\|A(t)\| = 2\sup_k size : q_{kk}(t) size : -2L(t) < \infty$ for almost all $t \geq 0$. Hence the operator function $A(t)$ from l_1 into itself is bounded for almost all $t \geq 0$ and locally integrable on $[0, \infty)$. Therefore we can consider (1) as a differential equation in the space l_1 with bounded operator.

It is well known (see Daleckii and Krein, 2002) that the Cauchy problem for differential equation (1) has a unique solution for an arbitrary initial condition, and $\mathbf{p}(s) \in \Omega$ implies $\mathbf{p}(t) \in \Omega$ for $t \geq s \geq 0$.

Denote by $E(t,k) = E(X(t) size : X(0) = k)$ the conditional expected number of customers in the system at instant t, provided that initially (at instant $t = 0$) k customers were present in the system.

Recall that a Markov chain $\{X(t),\ t \geq 0\}$ is called *weakly ergodic*, if $\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \to 0$ as $t \to \infty$ for any initial conditions $\mathbf{p}^*(0)$ and $\mathbf{p}^{**}(0)$, where $\mathbf{p}^*(t)$ and $\mathbf{p}^{**}(t)$ are the corresponding solutions of (1); and *exponentially weakly ergodic* if the difference tends to zero exponentially fast. A Markov chain $\{X(t),\ t \geq 0\}$ has the limiting mean $\varphi(t)$, if $\lim_{t\to\infty}(\varphi(t) - E(t,k)) = 0$ for any k.

3. Main bounds

Rewrite the forward Kolmogorov system (1) as

$$\frac{d\mathbf{p}}{dt} = A^*\left(t\right)\mathbf{p} + \mathbf{g}\left(t\right), \quad t \geq 0. \tag{2}$$

Here $\mathbf{g}\left(t\right) = \left(\beta_*\left(t\right), 0, 0, \ldots\right)^T$, $A^*\left(t\right) = \left(a^*_{ij}\left(t\right)\right)_{i,j=0}^{\infty}$, and

$$a^*_{ij}\left(t\right) = \begin{cases} q_{00}\left(t\right) - \beta_*\left(t\right), & \text{if } i = j = 0, \\ \mu_{j0}\left(t\right) + \beta_j(t) - \beta_*\left(t\right), & \text{if } i = 0,\ j > 0 \\ q_{ji}\left(t\right), & \text{otherwise}, \end{cases}$$

where $\beta_*\left(t\right) = \inf_i \beta_i\left(t\right)$.

Denote $\mathbf{y} = \mathbf{p}^* - \mathbf{p}^{**}$.
Then

$$\frac{d\mathbf{y}(t)}{dt} = A^*(t)\mathbf{y}(t).$$

Let now d_k be positive numbers for $k \geq 0$, and let

$$d = \inf_k d_k > 0, \quad d^* = \sup_k d_k \leq \infty.$$

Put $w_k = d_k y_k$, for $k \geq 0$. Consider a new vector function $\mathbf{w}(t) = D\mathbf{y}(t)$, where D is a diagonal matrix with entries d_k. Then we obtain

$$\frac{d\mathbf{w}(t)}{dt} = A_D^*(t)\mathbf{w}(t),$$

where $A_D^*(t) = DA^*(t)D^{-1} = \left(a^*_{i,j,D}(t)\right)_{i,j=0}^{\infty}$, with the corresponding elements.

Let

$$\beta_{**}(t) = \inf_i \left(|a^*_{i,i,D}(t)| - \sum_{j\neq i} a^*_{j,i,D}(t) \right), \tag{3}$$

Then one can write the following estimate for the upper right derivative of $\|\mathbf{w}(t)\|$

$$\frac{d_r^+}{dt}\|\mathbf{w}(t)\| \leq -\beta_{**}(t)\|\mathbf{w}(t)\|,$$

and then, dividing by the $\|\mathbf{w}(t)\|$ and integrating, one will have the following upper bound:

$$\|\mathbf{w}(t)\| \leq e^{-\int_0^t \beta_{**}(u)\,du}\|\mathbf{w}(0)\|.$$

Remark. In fact, there is also the usual right-hand derivative of the norm, this is the logarithmic norm, which we most often use (see for instance Zeifman et al., 2021b), so its application would lead to the same result.

If we compare different norms of the vector, we get

$$\|\mathbf{w}(t)\| = \|D\mathbf{y}(t)\| = \|D\left(\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\right)\|,$$

and

$$d\|\mathbf{y}(t)\| \leq \|\mathbf{w}(t)\| \leq d^*\|\mathbf{y}(t)\|.$$

Hence we have the following statement.

Theorem 1. *Let there exist a sequence* $\{d_k,\ k \geq 0\}$ *such that*

$$\int_0^{\infty} \beta_{**}(t)\, dt = +\infty. \tag{4}$$

Then the queue-length process $X(t)$ *is weakly ergodic and the following bound on the rate of convergence holds:*

$$\left\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\right\| \leq d^{-1}e^{-\int_0^t \beta_{**}(\tau)\,d\tau}\left\|D\left(\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\right)\right\|. \tag{5}$$

Moreover,
(i) if $d^* < \infty$ *then* $X(t)$ *is weakly ergodic in the uniform operator topology and the following bound holds*

$$\left\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\right\| \leq \frac{2d^*}{d}e^{-\int_0^t \beta_{**}(\tau)\,d\tau}, \tag{6}$$

for any initial conditions $\mathbf{p}^*(0), \mathbf{p}^{**}(0)$ *and any* $t \geq 0$.
(ii) if $d^* = \infty$, *and in addition* $W = \inf_{i\geq 1}\frac{d_i}{i} > 0$, *then* $X(t)$ *has the limiting mean, say* ﬃ$(t) = E(t,0)$, *and the following bound holds:*

$$|E(t,j) - E(t,0)| \leq \frac{d_0 + d_j}{W}e^{-\int_0^t \beta_{**}(\tau)\,d\tau}, \tag{7}$$

for any j *and any* $t \geq 0$.

Let now there exist positive R_{**} and b_{**} such that

$$e^{-\int_s^t \beta_{**}(\tau)\,d\tau} \leq R_{**}e^{-b_{**}(t-s)}, \tag{8}$$

for any $0 \le s \le t$. Then $X(t)$ is exponentially weakly ergodic and we can estimate the limiting regime itself by the following way.

Let, in addition the 'common catastrophe rate' $\beta_*(t)$ be bounded, i.e.

$$\beta_*(t) \le b^* < \infty \text{ for almost all } t \ge 0. \tag{9}$$

Denote by $U(t,s)$ the Cauchy operator of Eq. (2), then the solution of this equation looks as

$$\mathbf{p}(t) = U(t,0)\mathbf{p}(0) + \int_0^t U(t,\tau)\mathbf{g}(\tau)\,d\tau,$$

where $\mathbf{p}(0)$ is the initial probability distribution of $X(t)$ (initial condition).

Hence we have (in $1D$ norm, where $\|\mathbf{z}\|_{1D} = \|D\mathbf{z}\|$ and $\|B\|_{1D} = \|DBD^{-1}\|_1$):

$$\|\mathbf{p}(t)\|_{1D} \le \|U(t,0)\|_{1D}\|\mathbf{p}(0)\|_{1D} + \int_0^t \|U(t,\tau)\|_{1D}\|\mathbf{g}(\tau)\|_{1D}\,d\tau$$

$$\le R_{**}e^{-b_{**}t}\|\mathbf{p}(0)\|_{1D} + \int_0^t R_{**}e^{-b_{**}(t-\tau)}d_0 b^*\,d\tau \le o(1) + \frac{R_{**}d_0 b^*}{b_{**}},$$

and the following statement.

Theorem 2. *Let for some positive sequence $\{d_k\}$ inequalities (8) and (9) hold.*

Then the existing by Theorem 1 limiting regime satisfied the following bound:

$$\limsup_{t\to\infty} \|\mathbf{p}(t)\|_{1D} \le \frac{R_{**}d_0 b^*}{b_{**}}. \tag{10}$$

Example. Consider here as an example the model from Marin and Rossi (2020) and Zeifman et al. (2021a) with additional disasters (catastrophes).

Then the corresponding intensity matrix $A(t)$ of $X(t)$ has the following form:

$$A(t) = \begin{pmatrix} -\lambda(t) & \mu(t)+\gamma_1(t) & \gamma_2(t) & \gamma_3(t) & \cdots \\ \lambda(t)b_1 & -(\lambda(t)B_2+\mu(t)+\gamma_1(t)) & \mu(t) & 0 & \cdots \\ \lambda(t)b_2 & \lambda(t)b_2 & -(\lambda(t)B_3+\mu(t)+\gamma_2(t)) & \mu(t) & \cdots \\ \lambda(t)b_3 & \lambda(t)b_3 & \lambda(t)b_3 & -(\lambda(t)B_4+\mu(t)+\gamma_3(t))\ldots & \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix}$$

where $B_k = \sum_{i\ge k} b_i$, $B_1 = 1$, all $b_k \ge 0$, and $\sum_k kB_k < \infty$.

In the previous studies the authors suppose geometric decreasing of b_k. Here we outline the simple situation of slowly decreasing arrivals intensities, namely let $b_k = \frac{4}{k(k+1)(k+2)}$. Put $d_0 = 1$, $d_k = k+1$ for $k \ge 1$, and $\beta_*(t) = \inf_{i\ge1}\gamma_i(t)$. Then one has in (3)

$$\beta_{**}(t) = \beta_*(t) - \lambda(t)\sum_{k\ge1}(d_k-1)\,b_k = \beta_*(t) - \frac{1}{2}\lambda(t).$$

In particular, if $\lambda(t) = 2+2\cos 2\pi t$, $\gamma_k(t) = 2+\frac{1+\sin 2\pi t}{k}$, and $\mu(t)$ is an arbitrary 1-periodic function, that is, a function periodic in t with period equal to 1. Then $\beta_{**}(t) = 1+2\cos 2\pi t$.

Hence we have in $R_{**} \le 2$, $b_{**} = 1$, $b^* = 4$ in (8) and (9) respectively. Moreover $W = 1$, $X(t)$ is exponentially ergodic and has the limiting mean. Theorems 1 and 2 give us the following bounds:

$$\left\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\right\| \le 2e^{-t}\left\|D\left(\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\right)\right\|,$$

$$|E(t,j) - E(t,0)| \le 2(1+j)\,e^{-t},$$

and

$$\limsup_{t\to\infty}\|\mathbf{p}(t)\|_{1D} = \limsup_{t\to\infty} E(t,0) \le 8.$$

Acknowledgments

This research was supported by Russian Science Foundation under grant 19-11-00020. Author thanks the referees for helpful remarks.

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