Upper Bounds for the Rate of Convergence of Inhomogeneous Birth and Death Processes on $\mathbb{Z}$

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Abstract. Consideration is given to the unrestricted one-dimensional birth and death process on the integers. The process may take only unit steps to the right and left. The transition rates are assumed to be uniformly bounded, locally integrable non-random functions of time, and are allowed to depend on the state of the process. The approach based on the logarithmic norm is briefly described, which allows one to obtain the bounds on the rate of convergence to the limiting regime, whenever it exists.

INTRODUCTION

Restricted and unrestricted one-dimensional inhomogeneous continuous-time birth and death processes (BDP) have been the subject of extensive research during the past decades. Now it is quite a well-developed field of study containing various exact results, analytical and numerical methods, and open problems (see the reviews, for example, in the latest papers [1, Section 1] and [2], and references therein). One of the most typical settings (both for homogeneous and inhomogeneous cases), under which the analysis is performed, is when a Markov chain has a countable set of states (usually labelled with integers) and one of the endpoints of the set is held fixed. For example, if $X(t)$ denotes the queue size at the instant t in the classical $M_t/M_t/1/\infty$ queueing system, then $X(t) \in \{0, 1, \dots\}$; if $X(t)$ is the stock level in an inventory system [3], then $X(t)$ may be allowed to take any non-negative integer value as well as a finite number of negative integer values. The state spaces of these two one-dimensional BDPs have fixed left endpoints. By no means difficult is to think out the example of a one-dimensional BDP with countably infinite number of states and the state space being such, that its right endpoint is fixed. Less typical (but well-known, see [4, 5]) setting is the one, when the state space of the BPD is unrestricted in both directions. Queueing systems, known in the operations research community as double-ended queues (see [1, 6]), deliver an intuitive appealing example. In the simplest case the are two queues of infinite capacity (say I and II), running in parallel, each with a dedicated arrival ordinary flow of customers. Customers from both queues are served one-by-one (say in FIFO order) by the single server under the following restrictions: (i) the service takes negligible amount of time and (ii) in order to perform this service the server must take one customer from each queue. Whenever one (or both) of the queues is empty, the server remains idle. If $X(t)$ denotes the difference between the size of queue I and the size of queue II at time t, then we have the unit step, linear, unrestricted random walk on integers and $X(t)$ is its position at time t. Another, somewhat artificial, example can be extracted from the Markov predator-prey models (or other models of species coexistence), in which $X(t)$ is the difference between the predator and prey populations. The extent to which the analysis of such random walks can be performed heavily depends on the underlying assumptions. For various unrestricted BDPs with time-independent transition rates a plenty of results are available out there. The cases, when all the rates are non-random functions of time, are, as usual, less studied (for a short review here one can refer, for example, to [8]).

The purpose of this short note is to present some new analytic results, which complement the available research results in latter direction. Specifically the general approach is presented, which allows one to obtain (under certain assumptions on the transition rates) upper bounds[1] for the distance between two probability distributions of a inhomogeneous BDP with possibly state-dependent transition rates (in some special norms).

PRELIMINARIES

Let $X(t)$ be a BDP with the state space $\{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$ and the transition intensities $\mu_k(t)$ and $\lambda_k(t)$, $k \in \mathbb{Z}$. The transition diagram of $X(t)$ is shown in the figure below. In order to keep the figure and the matrices readable, whenever it does not introduce any ambiguity, the argument t of the intensity functions is omitted.

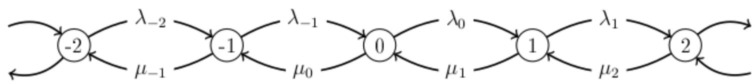

Figure 1. The transition diagram for $X(t)$

The transition intensities are allowed to be non-random arbitrary (not necessarily monotone) functions of time t, locally integrable on $[0, \infty)$, which may depend on the state of the BDP but must be bounded in the following sense: there exists a constant $L > 0$, such that for all $k \in \mathbb{Z}$ and $t \geq 0$

$$\lambda_k(t) \leq L, \ \mu_k(t) \leq L.$$

Let $p_k(t) = P\{X(t) = k\}$ be the probability that $X(t)$ is in state k at time t. Given any proper initial condition, the probabilistic dynamics of $X(t)$ is given by the forward Kolmogorov system of differential equations:

$$\frac{d}{dt}p_k(t) = \lambda_{k-1}(t)p_{k-1}(t) - (\lambda_k(t) + \mu_k(t))\, p_k(t) + \mu_{k+1}(t)p_{k+1}(t), \ k \in \mathbb{Z}.$$

or, in the vector form,

$$\frac{d}{dt}\mathbf{p}(t) = A(t)\mathbf{p}(t), \tag{1}$$

where $\mathbf{p}(t) = (\ldots, p_{-1}(t), p_0(t), p_1(t), \ldots)^T$ and $A(t)$ is the transposed intensity operator.

It turns out that using only the system (1), special linear transformations and the notion of logarithmic norm, it is possible to find the upper bounds for the rate of convergence of $\{p_k(t), k \in \mathbb{Z}\}$ to its limiting distribution, assuming that it exists.

In what follows $\|\cdot\|$ denotes the l_1-norm. For a linear transformation given by the infinite-dimensional matrix, say D, $\|\cdot\|_{1D}$ denotes the $1D$-norm, i.e. if $\mathbf{x}$ is an infinite dimensional column vector then $\|\mathbf{x}\|_{1D} = \|D\mathbf{x}\|$.

DESCRIPTION OF THE APPROACH

The first step is to transform the system of differential equations (1), so that eventually the state $\{0\}$ is removed from the consideration. This technique is well-known and described, for example, in [9, 10]. Since $p_0(t) = 1 - \sum\limits_{\substack{k=-\infty \\ k\neq 0}}^{\infty} p_k(t)$, the system (1) can be re-written as:

$$\frac{d}{dt}\mathbf{z}(t) = B(t)\mathbf{z}(t) + \mathbf{f}(t),$$

where

$$\mathbf{f}(t) = (\ldots, 0, \mu_0(t), \lambda_0(t), 0, \ldots)^T, \ \ \mathbf{z}(t) = (\ldots, p_{-2}(t), p_{-1}(t), p_1(t), p_2(t), \ldots)^T,$$

[1]The usefulness of such bounds is discussed elsewhere (see, for example, [7]).

and the linear operator $B(t)$ has the form

$$
B(t) = \begin{array}{c} \\ \\ -2 \\ -1 \\ 1 \\ 2 \\ \\ \end{array}
\begin{array}{c}
\begin{array}{cccccc} & -2 & -1 & 1 & 2 & \end{array} \\
\begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & -\lambda_{-2}-\mu_{-2} & \mu_{-1} & 0 & 0 & \cdots \\
\cdots & \lambda_{-2}-\mu_{0} & -\mu_{-1}-\lambda_{-1}-\mu_{0} & -\mu_{0} & -\mu_{0} & \cdots \\
\cdots & -\lambda_{0} & -\lambda_{0} & -\mu_{1}-\lambda_{1}-\lambda_{0} & \mu_{2}-\lambda_{0} & \cdots \\
\cdots & 0 & 0 & \lambda_{1} & -\mu_{2}-\lambda_{2} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\end{array}.
$$

The second step is to choose the infinite sequence $\{d_k, k = \pm 1, \pm 2, \dots\}$ of positive numbers. Introduce the linear transformation D^{**}, given by the matrix $D^{**} = diag\,(\dots, d_{-2}, d_{-1}, d_1, d_2, \dots)$, and the linear transformation D^{*}, given by the matrix

$$
D^{*} = \begin{array}{c} \\ \\ -2 \\ -1 \\ 1 \\ 2 \\ 3 \\ \\ \end{array}
\begin{array}{c}
\begin{array}{ccccccc} & -2 & -1 & 1 & 2 & 3 & \end{array} \\
\begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & -1 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & -1 & -1 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 1 & 1 & 1 & \cdots \\
\cdots & 0 & 0 & 0 & 1 & 1 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 1 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\end{array},
$$

Put $D = D^{**}D^{*}$. It is straightforward to check, that the linear transformation DBD^{-1}, with D^{-1} being the right inverse operator of D, is given by the matrix

$$
DBD^{-1} = \begin{array}{c} \\ \\ -4 \\ -3 \\ -2 \\ -1 \\ 1 \\ 2 \\ 3 \\ \\ \end{array}
\begin{array}{c}
\begin{array}{ccccccc} & -3 & -2 & -1 & 1 & 2 & \end{array} \\
\begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \mu_{-3}\frac{d_{-4}}{d_{-3}} & 0 & 0 & 0 & 0 & \cdots \\
\cdots & -\mu_{-2}-\lambda_{-3} & \mu_{-2}\frac{d_{-3}}{d_{-2}} & 0 & 0 & 0 & \cdots \\
\cdots & \lambda_{-2}\frac{d_{-2}}{d_{-3}} & -\mu_{-1}-\lambda_{-2} & \mu_{-1}\frac{d_{-2}}{d_{-1}} & 0 & 0 & \cdots \\
\cdots & 0 & \lambda_{-1}\frac{d_{-1}}{d_{-2}} & -\mu_{0}-\lambda_{-1} & \mu_{0}\frac{d_{-1}}{d_{1}} & 0 & \cdots \\
\cdots & 0 & 0 & \lambda_{0}\frac{d_{1}}{d_{-1}} & -\mu_{1}-\lambda_{0} & \mu_{1}\frac{d_{1}}{d_{2}} & \cdots \\
\cdots & 0 & 0 & 0 & \lambda_{1}\frac{d_{2}}{d_{1}} & -\mu_{2}-\lambda_{1} & \cdots \\
\cdots & 0 & 0 & 0 & 0 & \lambda_{2}\frac{d_{3}}{d_{2}} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdot & \cdots
\end{pmatrix}
\end{array}
$$

The third step is to apply the method of the logarithmic norm (see, for example, [10]), from which the ergodicity bounds for $\{p_k(t), k \in \mathbb{Z}\}$ follow. Denote by $\alpha_k(t)$ the sum of all elements in the kth column of DBD^{-1}. By direct inspection it can be checked that

$$
-\alpha_k(t) = \begin{cases}
\lambda_k(t) + \mu_{k+1}(t) - \frac{d_{k+1}}{d_k}\lambda_{k+1}(t) - \frac{d_{k-1}}{d_k}\mu_k(t), & k < -1 \\
\lambda_{-1}(t) + \mu_0(t) - \frac{d_1}{d_{-1}}\lambda_0(t) - \frac{d_{-2}}{d_{-1}}\mu_{-1}(t), & k = -1 \\
\lambda_0(t) + \mu_1(t) - \frac{d_2}{d_1}\lambda_1(t) - \frac{d_{-1}}{d_1}\mu_0(t), & k = 1 \\
\lambda_{k-1}(t) + \mu_k(t) - \frac{d_{k+1}}{d_k}\lambda_k(t) - \frac{d_{k-1}}{d_k}\mu_{k-1}(t), & k > 1.
\end{cases}
$$

Put $\alpha(t) = \inf_{k \neq 0} -\alpha_k(t)$. If $\int_0^\infty \alpha(t)\,dt = +\infty$, then $X(t)$ is weakly ergodic in $1D$-norm and for any $t \geq 0$ and any initial conditions $\mathbf{p}^*(0)$ and $\mathbf{p}^{**}(0)$ it holds that

$$
\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1D} \leq e^{-\int_0^t \alpha(\tau)\,d\tau}\,\|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\|_{1D}\,.
$$

Using the outlined approach one can also obtain some explicit estimates for the probabilities $\limsup_{t\to\infty} \mathsf{P}\{|X(t)| < N\}$ for $N > 0$ (if they exist) and two-sided truncations bounds, which allows to perform the numerical solution of (1) with the required accuracy.

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