Calculation of Mutual Capacitances for System of Conductors in Inhomogeneous Dielectric Media

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Abstract. Some new stochastic algorithms are suggested for capacitance extraction of 3-D conductor systems in inhomogeneous dielectric media.

INTRODUTION

The problems of finding potentials and mutual capacitances for complex three-dimensional objects have become widespread with the development of high-frequency electrical engineering. For computing capacitances in complicated three-dimensional geometries, usually, boundary-element technique use or Monte Carlo method (see, for example [1, 2]). In paper [2] Monte Carlo method is applied for conductors with plane boundary in homogeneous medium. The boundary-element technique applied to solve the system of potential theory integral equations. Monte Carlo method applied to solve boundary value problem (BVP). Modern Monte Carlo methods for solving boundary value problems are based on mean value formulas and do not require discretization of the problem. They are meshless. Consequently, there is no approximation error in them. Due to this, it is possible to estimate the error of the approximate solution of the problem during the calculations. Monte Carlo algorithms for a boundary value problem are based on the representation of its solution in the form of the mathematical expectation of some random variable, which in mathematical statistics is called an unbiased estimator. Usually the estimator is constructed on the trajectories of a homogeneous Markov chain (random walk). In our previous works, we developed algorithms for calculating of capacitances for a homogeneous medium on trajectories of a walk on spheres [3] and an inhomogeneous medium on trajectories of a walk on hemispheres [4], when dielectric interfaces are polyhedra. In this paper, we consider a new version of the hemisphere walk and its application to the calculation of electrostatic capacitances for various dielectric interfaces.

PROBLEM FORMULATION

In inhomogeneous media with permittivity $\varepsilon(x)$, the electrostatic potential $\varphi(x)$ satisfy the boundary value problem:

$$\left.\begin{array}{l}\Delta\varphi = 0,\ x \in R^3 \setminus (\Gamma_i \cup \Gamma_d); \varphi(x) \underset{\|x\|\to\infty}{\longrightarrow} 0; \varphi|_{\Gamma_i} = \varphi_i,\ \varphi_i = const; \varphi^+(x) = \varphi^-(x),\ x \in \Gamma_d;\\ \varepsilon^+ \frac{\partial\varphi^+(x)}{\partial n} = \varepsilon^- \frac{\partial\varphi^-(x)}{\partial n},\ x \in \Gamma_d;\\ \oint_{\Gamma_i} \varepsilon \frac{\partial\varphi}{\partial n} dS = -4\pi q_i.\end{array}\right\} \tag{1}$$

Here Γ_i denotes the conductor surfaces, Γ_d is the union of the dielectric interfaces, n is the external normal to Γ_d, φ^+ and φ^- are the values of the potential on different sides dielectric interfaces, ε^+ and ε^- are the permittivity constants on different sides dielectric interfaces, φ_i is the values of the potential on the Γ_i, and q_i is the charge on Γ_i.

Charges linearly depend on potentials [5]: $q_i = \sum_{j=1}^{m} C_{ij}\varphi_j$. Here, C_{ij} is mutual electrostatic capacitance for the conductors i and j. Hence, C_{ij} is equal to the charge q_i, when all potential $\varphi_k = 0$, if $k \neq j$, and $\varphi_j = 1$. Using Gauss's theorem and replacing the normal derivative $\partial\varphi/\partial n$ by its integral representation [3] in the ball, which lies entirely in the region with the dielectric constant ε_i, we obtain an integral representation of the mutual capacitance of the i-th and j-th conductors:

$$C_{ij} = -\frac{1}{\sigma_\Gamma} \oint_\Gamma \frac{\varepsilon}{4\pi r^2} \oint_{S_r} \frac{3\sigma_\Gamma}{4\pi r^2} (y - x, n)\varphi(y) d_y S\, d_x S, \tag{2}$$

where Γ is the surface containing the i-th conductor inside and separating it from others conductors and interfaces; x is a point on the shell around the i-th conductor; r is distance from point x to the nearest conductor or interface; S_r is sphere of radius r centered at point x; σ_Γ is a surface area Γ; y is a point on S_r

Random Walk and Estimators

Using formula (2) we have unbiased estimator

$$\xi = \frac{3\varepsilon(X)\sigma_\Gamma}{4\pi r} (\omega, n)\varphi(X + r\omega) \tag{3}$$

for capacitance C_{ij}. Here, a random point X is uniformly distributed on Γ, $r = r(X)$, and ω is an isotropic vector (random unit vector). It remains to estimate the potential at the point $Y = X + r(x)\omega$. This can be done using the mean value formulas

$$\varphi(x) = \int_Q \varphi(y) P(x, dy), x \in Q, \tag{4}$$

where $Q = \mathbf{R}^3 \setminus D$, and D is the set of interior points of all conductors. The unbiased estimators for $\varphi(Y)$ are constructed on trajectories of Random Walk $\{Y_k\}_{k=0}^{\infty}$, $(Y_0 = Y)$, in space Q. The kernel $P(x, dy)$ must be stochastic or sub-stochastic. It determines the distribution of the next point of the Random Walk over the current point. Let

$$\xi_0 = \frac{3\varepsilon(X)\sigma_\Gamma}{4\pi r} (\omega, n).$$

If at time k the "weight" $W_k = P(Y_k, Q) < 1$, then the current value of the estimator is multiplied by the "weight": $\xi_{k+1} = W_k \xi_k$. The Random Walk stops at time ν, when it reaches the δ-boundary of the conductors, that is, when the distance $\mathrm{dist}(Y_k, \partial\overline{D})$ from point Y_k to the boundary of the conductors becomes less than the δ. Hence, it must satisfy the condition $P\{\nu < \infty\} = 1$. We define estimator $\xi_\delta = \xi_\nu$, if $\mathrm{dist}(Y_\nu, \Gamma_j) < \delta$, and zero, otherwise. If the boundary ∂D is smooth enough, then $|C_{ij} - E\xi_\delta| < c\delta$, for some constant c. In practice, the estimator ξ_δ is simulated in a reasonable time, only if $E\xi_\nu < \infty$. Having received a sufficient number of realizations of the estimator ξ_δ and calculating their arithmetic mean, we obtain an approximate value of the capacitance C_{ij}.

We will now describe some of the types of walks used to calculate the capacitances of conductors.

Random Walk on Spheres

Random Walk on Spheres is used to solve the external Dirichlet problem for the Laplace equation (see [**?**]) and allows you to calculate the capacitances of conductors in a homogeneous medium [3]. Let all the conductors lie inside a sphere S_R of radius R centered at the origin, $\{\omega_k\}_{k=1}^{\infty}$ be a sequence of independent isotropic vectors. If $|Y_k| \le R$, then $Y_{k+1} = r_k \omega_{k+1}$, where $r_k = \mathrm{dist}(Y_k, \overline{D})$. Otherwise, $\varphi(Y_K)$ is calculated by the Poisson formula, "weight" $W_k = R/|Y_k|$, and Y_{k+1} is distributed on the sphere S_R with density

$$p(Y_k, y) = \frac{|Y_k|^2 - R^2}{|Y_k - y|^3} \cdot \frac{|Y_k|}{4\pi R^2}. \tag{5}$$

Random Walk on Hemispheres

Random Walk on Hemispheres allows you to calculate capacitances, when dielectric interfaces are polyhedra [4]. Let all the conductors and dielectric interfaces lie inside a sphere S_R of radius R centered at the origin. If $|Y_k| > R$, then $Y_{k+1} \in S_R$ and has a distribution density (5). "Weight" $W_k = R/|Y_k|$. Now, let $Y_k \in \Gamma_{d_l}$, where Γ_{d_l} is a component of the dielectric interface. Next, we choose the maximum r, such, that $0 < r < \mathrm{dist}(Y_k, \overline{D} \cup \Gamma_d \setminus \Gamma_{d_l})$ and part of the Γ_{d_l}, lying in sphere $S_r(Y_k)$, is plane. The sphere is divided into two parts $S_r^+(Y_k)$ and $S_r^-(Y_k)$ lying in media with permittivity constants ε^+ and ε^-, respectively. The point Y_{k+1} is uniformly distributed in $S_r^+(Y_k)$ or in $S_r^-(Y_k)$ with probability

$\varepsilon^+/(\varepsilon^+ + \varepsilon^-)$ and $\varepsilon^-/(\varepsilon^+ + \varepsilon^-)$ respectively. If $Y_k \notin \Gamma_d$ and $|Y_k| \le R$, then Y_{k+1} is distributed on a sphere or hemisphere. The $\widehat{Y_k}$ center of the hemisphere must be in a plane containing a face of the conductor surface or interface and $\widehat{Y_k}$ is the orthogonal projection of Y_k onto this plane. Hemisphere radius $r_k = |Y_k - \widehat{Y_k}|/\beta$, where $0 < \beta < 1$ is a fixed constant. The hemisphere must be contained in a medium with a dielectric constant $\varepsilon(Y_k)$. The distribution density of the point Y_{k+1} on the hemisphere is the normal derivative of the Green's function for the half of the ball. If it is impossible to construct such a hemisphere, then Y_{k+1} is distributed uniformly on a sphere of radius $r_k = \text{dist}(Y_k, \overline{D} \cup \Gamma_d)$, centered at Y_k.

New Random Walk on Hemispheres

Let us extend the RWH algorithm to the case when the dielectric interfaces are convex. Let γ be a connected convex part of some dielectric interface Γ_{d_l} lying inside a sphere $S_r(x)$ of radius r, centered at point x. For all $y \in \Gamma_{d_l}$ we choose the direction of the normal vector n_y so, that the surface Γ_{d_l} lies in the half-space $(z - y, n_y) \le 0$. Surface γ divides the sphere into two parts S^+ and S^-, lying in media with permittivity constants ε^+ and ε^-, respectively, and $(z - y, n_y) \le 0$ for all $z \in S^-$. For the harmonic function $\varphi(x)$, the mean value formulas are valid:

$$\varphi(x) = \frac{1}{1+\lambda} \cdot \frac{1}{2\pi r^2} \int_{S^-} \varphi(y) d_y S + \frac{\lambda}{1+\lambda} \cdot \frac{1}{2\pi r^2} \int_{S^+} \varphi(y) d_y S + \frac{1-\lambda}{1+\lambda} \cdot \frac{1}{2\pi} \int_\gamma \frac{\cos\varphi_{xy}}{|x-y|^2} \varphi(y) d_y S, \quad x \in \gamma, \tag{6}$$

$$\varphi(x) = \frac{1}{4\pi r^2} \int_{S^-} \varphi(y) d_y S + \lambda \cdot \frac{1}{4\pi r^2} \int_{S^+} \varphi(y) d_y S + (1-\lambda) \cdot \frac{1}{4\pi} \int_\gamma \frac{\cos\varphi_{xy}}{|x-y|^2} \varphi(y) d_y S, \quad x \notin \gamma, \quad \varepsilon(x) = \varepsilon^-, \tag{7}$$

$$\varphi(x) = \frac{1}{4\pi r^2} \int_{S^+} \varphi(y) d_y S + \frac{1}{\lambda} \cdot \frac{1}{4\pi r^2} \int_{S^-} \varphi(y) d_y S - \left(1 - \frac{1}{\lambda}\right) \cdot \frac{1}{4\pi} \int_\gamma \frac{\cos\varphi_{xy}}{|x-y|^2} \varphi(y) d_y S, \quad x \notin \gamma, \quad \varepsilon(x) = \varepsilon^+, \tag{8}$$

where $\lambda = \varepsilon^+/\varepsilon^-$ and φ_{xy} — angle between vectors n_y, $y - x$.

If $\lambda < 1$, the formula (6) defines the stochastic kernel. To simulate the transition from the surface γ, we simulate with the probability $\lambda/(1+\lambda)$ a random direction ω, that satisfies the condition $(\omega, n_x) > 0$, and define $Y = x + r\omega$. With probability $1/(1+\lambda)$, we simulate a random direction ω satisfying the condition $(\omega, n_x) < 0$. We calculate $Y = x + r\omega$. If $Y \notin S^-$, we change Y to a point $Z \in \gamma$, which is visible from x in the direction ω.

If $\lambda < 1$, the formula (7) defines the stochastic kernel also. To simulate the transition from x, we simulate a random direction ω and calculate $Y = x + r\omega$. If $Y \notin S^-$, than with probability $1 - \lambda$ we change Y to a point $Z \in \gamma$, which is visible from x in the direction ω.

If $\lambda > 1$, the formula (8) defines the stochastic kernel, if any ray outgoing from point x intersects γ at no more than one point. The modeling procedure is similar to the algorithm for the formula (7).

Thus, formulas (6–8) make it possible to simulate transitions from a region with a higher dielectric constant to a region with a lower dielectric constant. To pass from point x through the interface Γ_{d_l}, it is sufficient to take such $r \le \text{dist}(x, \overline{D} \cup \Gamma_d \setminus \Gamma_{d_l})$ that $S_r(x) \cap \Gamma_{d_l} \ne \emptyset$. Reverse transitions can be provided using, for example, formulas for solving external and internal Dirichlet problems for standard domains. The exit from the "bad" point x can be done by Random Walk on Spheres or Hemispheres in the set $Q(x) = \{y | \varepsilon(y) = \varepsilon(x)\}$. As always, from distant points of the external medium there is a transition to the sphere S_R.

NUMERICAL RESULTS

Coated Sphere

Capacity of single conductive sphere with radius a in concentric dielectric shell with radius b and permittivity ε and free space permittivity 1, could be calculated using formula $4\pi\varepsilon_0\varepsilon ab/(\varepsilon a + b - a)$ in [5]. Results of estimation with "New Random Walk on Hemispheres" (NRWH), NRWH error estimation (NRWHEE) and FastCap2 (sphere refinement depth 5, -t 0.001) are presented in Table 1.

Two Spheres in Dielectrical Shell

Mutual capacities of two conductive spheres in dielectrical shell were computed using FastCap2 (sphere refinement depth 5, -t 0.001) and NRWH. FastCap2 have some issues with high permittivities [1], so comparison was made only

TABLE 1. Capacity of coated sphere, $C/4\pi\varepsilon_0$

	Analytical	$FastCap2$	NRWH, 10^6	NRWHEE, 10^6	NRWH, 10^7	NRWHEE, 10^7
$a = 1, b = 3, \varepsilon = 2$	1.5000	1.5018	1.5075	$3.12 \cdot 10^{-2}$	1.5006	$9.8 \cdot 10^{-3}$
$a = 1, b = 3, \varepsilon = 10$	2.5000	2.5872	2.4979	$1.89 \cdot 10^{-1}$	2.5155	$5.97 \cdot 10^{-2}$
$a = 1, b = 3, \varepsilon = 100$	2.9411	4.3926	3.6100	2.05	3.3893	$6.50 \cdot 10^{-1}$

TABLE 2. Mutual capacities of two spheres, $C_{ij}/4\pi\varepsilon_0$

	$FastCap2$	NRWH, 10^6	NRWHEE, 10^6	NRWH, 10^7	NRWHEE, 10^7
$\varepsilon = 2$					
C_{11}	9.8444	9.8210	$1.599 \cdot 10^{-1}$	9.8732	$5.055 \cdot 10^{-2}$
C_{12}, C_{21}	-3.3396	-3.3541	$2.260 \cdot 10^{-2}$	-3.3356	$7.146 \cdot 10^{-3}$
C_{22}	6.0965	6.1114	$6.762 \cdot 10^{-2}$	6.0700	$2.136 \cdot 10^{-2}$
$\varepsilon = 10$					
C_{11}	33.262	31.174	$8.269 \cdot 10^{-1}$	32.585	$2.615 \cdot 10^{-1}$
C_{12}, C_{21}	-21.0378	-21.220	$1.320 \cdot 10^{-1}$	-21.128	$4.169 \cdot 10^{-2}$
C_{22}	25.3801	25.068	$3.495 \cdot 10^{-1}$	25.206	$1.105 \cdot 10^{-1}$

for permittivities up to 10. First sphere's center at $(1, 2, 3)$, radius — 5, second sphere's center at $(10, 3, 11)$, radius — 3, dielectric center — $(0, 0, 0)$, radius — 20, free space permittivity 1. Results of estimation with "New Random Walk on Hemispheres" (NRWH), NRWH error estimation (NRWHEE) and FastCap2 are presented in Table 2.

CONCLUSION

We developed some new numerical algorithms to extracting capacitances. Computer experiment shows that the algorithms are effective. For systems which the capacitances are calculated analytically [5], it is shown that the accuracy of the Monte Carlo approximation is within the statistical error. In more complex examples, the simulation results are compared with the results of calculating of the capacitances, using the FastCap2 program [6]. The algorithm also works correctly in the case when the ratio of the permittivities are 100 or more.

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